Step functions and temporal shifts

MATH 2250 Lecture 30
Book section 10.5

November 13, 2019

Step functions

The focus in this section is deriving and discussing one final Laplace transform property: temporal shifts.

To proceed, we need to introduce step functions.

Step functions

The focus in this section is deriving and discussing one final Laplace transform property: temporal shifts.

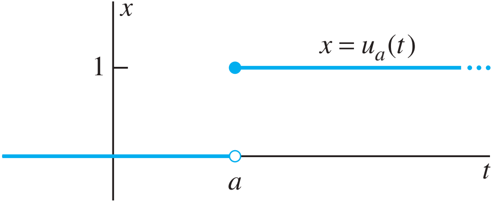
To proceed, we need to introduce step functions.

The **step function** or **Heaviside function** $u(t)$ is defined as

$$u(t) = \begin{cases} 0, & x < 0 \\ 1, & x \geqslant 0 \end{cases}$$

Note that shifts can place the discontinuity at aribitrary locations.

$$u_a(t) := u(t-a)$$



t Translation

Let $F(s)$ be the Laplace transform of $f(t)$ and a an arbitrary constant.

Then $e^{-sa}F(s)$ can be written using temporal shifts.

t Translation

Let $F(s)$ be the Laplace transform of $f(t)$ and a an arbitrary constant.

Then $e^{-sa}F(s)$ can be written using temporal shifts.

$$\begin{aligned} e^{-sa}F(s) &= e^{-sa}\int_0^\infty e^{-s\tau} f(\tau)\mathrm{d}\tau \\ &= \int_0^\infty e^{-s(\tau+a)} f(\tau)\mathrm{d}\tau \\ &\overset{t=\tau+a}{=} \int_a^\infty e^{-st} f(t-a)\mathrm{d}t \\ &= \int_0^\infty e^{-st} u(t-a) f(t-a)\mathrm{d}t \end{aligned}$$

Thus, "multiplication by e^{-sa} in s is shifting by a in t."

Examples

Example

Compute the inverse Laplace transform of $\frac{e^{-sa}}{s^n}$.

Examples

Example

Compute the inverse Laplace transform of $\frac{e^{-sa}}{s^n}$.

Example (Example 10.5.2)

Compute the Laplace transform of

$$g(t) = \begin{cases} 0, & t < 3 \\ t^2, & t \geqslant 3 \end{cases}$$

Examples

Example

Compute the inverse Laplace transform of $\frac{e^{-sa}}{s^n}$.

Example (Example 10.5.2)

Compute the Laplace transform of

$$g(t) = \begin{cases} 0, & t < 3 \\ t^2, & t \geqslant 3 \end{cases}$$

Example (Example 10.5.3)

Compute the Laplace transform of

$$f(t) = \begin{cases} \cos(2t), & 0 \leqslant t \leqslant 2\pi \\ 0, & \text{otherwise} \end{cases}$$

Discontinuous forcing

The main utility of this shift theorem for Laplace transforms is in solving DE's.

Example (Example 10.5.4)

Solve the initial value problem

$$x'' + 4x = f(t), \qquad\qquad x(0) = x'(0) = 0,$$

where $f(t)$ is as in the previous example:

$$f(t) = \begin{cases} \cos(2t), & 0 \leqslant t \leqslant 2\pi \\ 0, & \text{otherwise} \end{cases}$$

Summary of Laplace transform properties

t domain	s domain
$f(t)$	$F(s)$
$g(t)$	$G(s)$
$f(t)e^{-at}$	$F(s-a)$
$f(t-a)u(t-a)$	$F(s)e^{-sa}$
$(-t)^n f(t)$	$F^{(n)}(s)$
$f^{(n)}(t)$	$s^n F(s) - \sum_{j=0}^{n-1} f^{(j)}(0) s^{n-j}$
$(f * g)(t) = \int_0^t f(\tau) g(t-\tau) \mathrm{d}\tau$	$F(s)G(s)$