Laplace transform manipulations

MATH 2250 Lecture 28
Book section 10.3

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DE's and Laplace transforms

We have use Laplace transforms to solve DE's of the form

$$ax''(t) + bx'(t) + cx(t) = f(t),$$

Typically, this requires us to compute inverse transforms of rational functions,

$$X(s) = \frac{P(s)}{Q(s)},$$

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We discuss two helpful techniques for computing these inverse transforms:

- partial fraction decompositions
- shift properties of the Laplace transform

Partial fractions (1/2)

Partial fraction decompositions is a technique for transforming $P(s)/Q(s)$ into a sum of "simpler" rational functions:

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots$$

where $a_1, a_2, \ldots$ are the roots of Q.

The coefficients $A_1, A_2, \ldots$ are determined from P, Q, and a_j by some linear algebra.

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The technique is complicated only by the event that sometimes roots of Q have multiplicity greater than 1, e.g.:

$$\frac{P(s)}{Q(s)} = \frac{s}{(s-2)^2}$$

When a root a has multiplicity n, the correct partial fractions decomposition is

$$\frac{P(s)}{Q(s)} = \frac{P(s)}{(s-a)^n} = \sum_{j=1}^{n} \frac{A_j}{(s-a)^j}$$

Partial fractions (2/2)

Example

Compute the partial fractions decompositions of

$$F(s) = \frac{s+2}{(s-2)^2}, \qquad\qquad G(s) = \frac{s}{(s-1)^3}$$

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These factors can appear with higher multiplicities as well:

$$\frac{s}{s(s^2+4)^2} = \frac{A}{s} + \frac{Bs+C}{s^2+4} + \frac{Ds+E}{(s^2+4)^2}$$

Translations and shifts (1/3)

We will need a second skill: the ability to deal with *shifts*.

I.e., given $F(s)$ and its inverse Laplace transform $f(t)$, what is the inverse Laplace transform of $F(s-a)$?

Translations and shifts (1/3)

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I.e., given $F(s)$ and its inverse Laplace transform $f(t)$, what is the inverse Laplace transform of $F(s-a)$? A simple computation shows that if,

$$F(s) = \mathcal{L}\{f\} = \int_0^\infty f(t)e^{-st}\mathrm{d}t,$$

then

$$F(s-a) = \int_0^\infty f(t)e^{-(s-a)t}\mathrm{d}t = \int_0^\infty e^{at}f(t)e^{-st}\mathrm{d}t = \mathcal{L}\{f(t)e^{at}\}.$$

Thus, *shifts* in s-space correspond to *multiplication by exponentials* in t-space.

Examples (1/2)

Example (Example 10.3.2)

Compute the inverse Laplace transform of

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Example (Example 10.3.1)

Solve for the solution $x(t)$ from a mass-spring-damper system governed by the DE

$$x'' + 6x' + 34x = 0, \quad x(0) = 3, \;\; x'(0) = 1.$$

Examples (2/2)

Example (Example 10.3.4)

Solve for the solution $x(t)$ from a *forced* mass-spring-damper system governed by the DE

$$x'' + 6x' + 34x = 30\sin 2t, \quad x(0) = 0, \;\; x'(0) = 0.$$

Example (Example 10.3.5)

Compute the solution to the forced mass-spring system

$$x'' + \omega_0^2 x = F_0 \sin(\omega t), \quad x(0) = x'(0) = 0$$