DE's with Laplace transforms

MATH 2250 Lecture 27
Book section 10.2

November 1, 2019

DE's and Laplace transforms

Our goal is to solve DE's using Laplace transforms. Given,

$$ax''(t) + bx'(t) + cx(t) = f(t),$$

applying the Laplace transform yields

$$a\mathcal{L}\{x''\} + b\mathcal{L}\{x'\} + c\mathcal{L}\{x\} = \mathcal{L}\{f\}.$$

Thus, we ned to compute Laplace transforms of derivatives to proceed.

Laplace transforms of derivatives (1/2)

Suppose we are given $f(t)$. What is $\mathcal{L}\{f'(t)\}$?
By definition:

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Using integration by parts,

$$\begin{aligned} u &= e^{-st}, & v &= f(t), \\ u' &= -se^{-st}, & v' &= f'(t), \end{aligned}$$

we obtain

$$\mathcal{L}\{f'(t)\} = f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\mathrm{d}t.$$

Laplace transforms of derivatives (1/2)

Assume that $F(s) := \mathcal{L}\{f\}$ exists for $s > c$. Then $\mathcal{L}\{f'(t)\}$ is given by

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Note that this procedure can be iterated:

$$\begin{aligned} \mathcal{L}\{f''\} &= s\mathcal{L}\{f'\} - f'(0) \\ &= s\left[sF(s) - f(0)\right] - f'(0) \\ &= s^2 F(s) - sf(0) - f'(0) \end{aligned}$$

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And in general:

$$\begin{aligned}
\mathcal{L}\{f^{(n)}\} &= s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{n-2}(0) - f^{(n-1)}(0) \\
&= s^nF(s) - \sum_{j=0}^{n-1} s^{n-1-j}f^{(j)}(0)
\end{aligned}$$

Examples (1/2)

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Compute the Laplace transform of $f'(t)$, where $f(t) = \sin(at)$.

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Example (Example 10.2.1)

Solve the initial value problem

$$x'' - x' - 6x = 0, \qquad x(0) = 2, \;\; x'(0) = -1.$$

Examples (2/2)

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Example (Example 10.2.4)

Compute the Laplace transform of $f(t) = te^{at}$.

Example (Example 10.2.5)

Compute the Laplace transform of $f(t) = t\sin(kt)$.

The mathematical idea

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The price we pay: we must learn how to efficiently compute the correspondence $f(t) \leftrightarrow F(s)$.