# Forced oscillation and resonance

MATH 2250 Lecture 26
Book section 10.1

October 28, 2019

# Laplace transforms

We are comfortable solving some second-order constant coefficient equations:

$$x''(t) + a_1 x'(t) + a_0 x(t) = f(t),$$

but our success depends on the form of $f(t)$.

For example, if $f$ is discontinuous, we do not have a good way to solve this equation.

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The method of **Laplace transforms** is meant to address this deficiency. First couple of lectures: understand the transform.

Given $f(t)$ for $t \geqslant 0$, the Laplace transform of $f$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} =: \int_0^\infty e^{-st} f(t) \mathrm{d}t.$$

# Examples

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The simplest way to understand this is to use it.

## Example (Example 10.1.1)

Compute $F(s) = \mathcal{L}\{f\}$ for $f(t) = 1$.

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## Example (Example 10.1.2)

Compute $F(s) = \mathcal{L}\{f\}$ for $f(t) = e^{at}$ for some (possibly complex!) scalar $a$.

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# Laplace transforms and linearity (1/2)



A property of Laplace transforms that we will use extensively is *linearity*.

The Laplace transform $\mathcal{L}$ is a **linear** operator, i.e.,

$$\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$
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## Example

Compute $F(s) = \mathcal{L}\{f\}$ for $f(t) = \cos at$ for $a$ a real-valued scalar.

# Laplace transforms and linearity (2/2)



Example (Example 10.1.6)

Compute $F(s) = \mathcal{L}\{f\}$ for $f(t) = 3e^{2t} + 2\sin^2(3t)$.

# Inverse Laplace transforms

Laplace transforms have existence and uniqueness properties:

If $f(t)$ is piecewise continuous and satisfies

$$|f(t)| \leqslant Me^{ct}, \quad \text{for all } t \geqslant T,$$

for some $M$ and $c$, and $T$, then $F(s)$ exists and is unique for all $s > c$.

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## Example

Compute the inverse Laplace transform of $F(s) = \frac{1}{s}$ and $G(s) = \frac{s}{s^2+9}$ with $s > 0$.