Forced oscillation and resonance

MATH 2250 Lecture 25
Book section 5.6

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Forced oscillations

In systems whose homogeneous equations exhibit harmonic motion,

$$mx'' + cx' + kx = f(t),$$

the forcing function f is often an oscillatory function, e.g, $f(t) = \sin(\omega t)$ or $f(t) = \cos(\omega t)$.

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There are two regimes of interest: the undamped, and the damped case.

Undamped forced oscillations

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where

$$\omega_0 = \sqrt{\frac{k}{m}}, \qquad A = \frac{F_0/m}{\omega_0^2 - \omega^2}.$$

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If we further impose $x(0) = x'(0) = 0$, then we have

$$x(t) = -A \cos(\omega_0 t) + A \cos(\omega t),$$

$$x(t) = 2A \left(\sin \frac{1}{2}\omega_0 t\right) \left(\sin \frac{1}{2}\omega t\right)$$

Thus, the solution is a faster frequency modulated by a slower one.

Pure resonance

In our undamped forced oscillations case, if $\omega = \omega_0$, some coefficients blow up.
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The solution to the DE in this case is

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + c_3 t \cos(\omega_0 t) + c_4 t \sin(\omega_0 t).$$

Therefore, this model contains solutions whose amplitudes *grow* in time.

Generally speaking, this is interpreted as instability of a system.

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But the particular solution is one of the form $\cos(\omega t)$ and $\sin(\omega t)$ and thus are **steady periodic** components.

Thus, the full solution is a sum of transient and steady periodic components.

Practical resonance

$$mx'' + cx' + kx = F_0 \cos(\omega t),$$

Using undetermined coefficients, we can compute that the particular (steady periodic) solution is of the form

$$x_p(t) = C(\omega)\cos(\omega t - \alpha),$$

where

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Thus for $\omega > 0$ there is a maximum (but finite!) value of $C(\omega)$ occurs if $c < \sqrt{2km}$.

The ω that causes this maximum $C(\omega)$ results in **practical resonance**.