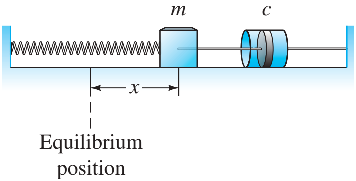
Mechanical Vibrations

MATH 2250 Lecture 23
Book section 5.4

October 21, 2019

Mass-spring-damper systems

Second-order constant coefficient DE's arise in common situations.



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Mass-spring-damper systems

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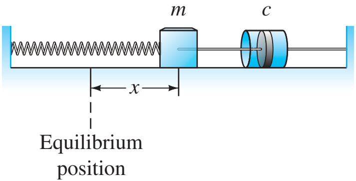

An object with mass m is attached to a spring with spring constant k, and to a damper ("dashpot") with constant c.

If $x(t)$ is the horizontal position of the object (relative to equilibrium), then Newton's second law yields:

$$mx''(t) = m\,a(t) = \sum(\text{forces}) = -kx - cv + F_{\text{ext}} = -kx - cx' + F_{\text{ext}}$$

Thus, the DE describing the motion of the block is:

$$mx'' + cx' + kx = F_{\text{ext}},$$

where m, c, and k are positive constants, F_{ext} is a given forcing function.

Mass-spring-damper characteristics

$$mx'' + cx' + kx = F_{\text{ext}},$$

If $c = 0$, the system is called **undamped**.
If $F_{\text{ext}} = 0$, the system is **free** or **unforced**.

Many physical models yield this equation: motions of pendulums, propagation of waves (water, air, music, seismic), quantum mechanics, ...

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Since all the constants are positive, we can deduce some properties of the homogeneous system from the characteristic equation roots:

$$r = \frac{1}{2m}\left[-c \pm \sqrt{c^2 - 4km}\right].$$

- If $c^2 - 4km < 0$: the real part of the roots are negative $\Longrightarrow$ exponential decay of solutions.
- If $c^2 - 4km > 0$: the roots are real, distinct, and negative $\Longrightarrow$ exponential decay of solutions.

Free, undamped motion

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The special case of $c = 0$ and $F_{\text{ext}} = 0$ comes up so often that it deserves special study.

This simplification results in **simple harmonic motion**.

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$$mx'' + kx = 0$$

The general solution to this equation is

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t), \qquad \omega := \sqrt{\frac{k}{m}}.$$

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When coupled with initial data, say $x(0) = x_0$ an $x'(0) = x_0'$, we obtain

$$x(t) = x_0 \cos(\omega t) + \frac{x_0'}{\omega} \sin(\omega t), \qquad \omega := \sqrt{\frac{k}{m}}.$$

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It is more instructive to write this as

$$x(t) = A \cos(\omega t - \phi),$$

which can be accomplished using trigonometric identities:

$$A = \sqrt{x_0^2 + \left(\frac{x_0'}{\omega}\right)^2}, \qquad \phi = \arctan\left(\frac{x_0'}{\omega x_0}\right).$$

Thus, this motion is **oscillatory**.

Free, undamped motion

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$$x(t) = A\cos(\omega t - \phi),$$

Properties of the trajectory $x(t)$:

- A is the **amplitude**.
- ω is the **frequency**.
- $\frac{2\pi}{\omega}$ is the **period**.
- ϕ is the **phase** or **lag**.

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Example (Example 5.4.1)

A body with mass $\frac{1}{2}$ kilogram (kg) is attached to the end of a spring that is stretched 2 meters (m) by a force of 100 newtons (N). It is set in motion with initial position $x_0 = 1$ (m) and initial velocity $v_0 = -5$ (m/s). (Note that these initial conditions indicate that the body is displaced to the right and is moving to the left at time $t = 0$.) Find the position function of the body as well as the amplitude, frequency, period of oscillation, and time lag of its motion.

Free, damped motion

Moving beyond simple harmonic motion, we consider the damped case.

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The general solution depends on the relationship between m, c, $k > 0$.

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- (**Underdamped**) If $c^2 - 4km < 0$:

  $$x(t) = C_1 \exp(-pt) \cos(\omega t) + C_2 \exp(-pt) \sin(\omega t),$$

  with $p := c/(2m) > 0$ and $\omega = \sqrt{4km - c^2}/(2m)$.

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  with

  $$p_1 = \frac{1}{2m}\left[-c + \sqrt{c^2 - 4km}\right], p_2 = \frac{1}{2m}\left[-c - \sqrt{c^2 - 4km}\right].$$

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- (**Criticially Damped**) If $c^2 - 4km = 0$:

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