Constant coefficient homogeneous equations

MATH 2250 Lecture 22
Book section 5.3

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Constant coefficient homogeneous equations

In this section we consider computing the general solution to the DE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

where $a_0, a_1, \ldots, a_n$ are constants, and $a_n \neq 0$.

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where $a_0, a_1, \ldots, a_n$ are constants, and $a_n \neq 0$.

As before, the way forward is to use the ansatz

$$y(x) = \exp(rx),$$

with r an unknown constant, in the DE. This yields the **characteristic equation**,

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = \sum_{j=0}^{n} a_j r^j = 0,$$

which is a condition on possible values of r.

Distinct, real-valued roots

If the characteristic equation has distinct, real-valued roots, $r = r_1, r_2, \ldots, r_n$, then we can identify n linearly independent solutions:

$$y_1(x) = \exp(r_1 x), \;\; y_2(x) = \exp(r_2 x), \;\; \cdots \;\; y_n(x) = \exp(r_n x).$$

Therefore, in this case of **distinct, real roots**, the general solution to the DE is

$$Y(x) = \sum_{j=1}^{n} c_j y_j(x).$$

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Example (Example 5.3.1)

Solve the initial value problem

$$y^{(3)} + 3y'' - 10y' = 0,$$
$$y(0) = 7, \;\; y'(0) = 0, \;\; y''(0) = 70.$$

Real, repeated roots

The case of real, repeated roots is more delicate. To simplify the situation, suppose we have a DE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

whose characteristic equation

$$\sum_{j=0}^{n} a_j r^j = 0,$$

has $k > 1$ repeated real roots at r_0. I.e., it can be written as

$$(r - r_0)^k \prod_{j=1}^{n-k} (r - r_j) = 0,$$

where $r_1, r_2, \ldots, r_{n-k}$ are distinct. The focus is on the repated root r_0.

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Can we find k linearly independent solutions from the repeated root? Clearly one solution is

$$u(x) = \exp(r_0 x),$$

but we need $k - 1$ more solutions.

Variation of parameters (1/3)

The strategy to compute the remaining solutions appeals to a technique called *variation of parameters*.

First note that the distinct roots are distractions. I.e., we can focus on finding solutions for a simpler characteristic equation and its associated DE:

$$(r - r_0)^k = 0 \;\Leftrightarrow\; \left[\frac{\mathrm{d}}{\mathrm{d}x} - r_0\right]^k y = 0$$

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The idea is as follows: we know that $u(x) = c\exp(r_0 x)$ solves the DE for a constant c.

A reasonable guess then, is that perhaps

$$w(x) = p(x)\exp(r_0 x),$$

for some unknown <u>function</u> $p(x)$, will solve the equation. What should p satisfy?

Variation of parameters (2/3)

Note that

$$\begin{aligned}\left[\frac{\mathrm{d}}{\mathrm{d}x} - r_0\right] w(x) &= p'(x)\exp(r_0 x) + r_0 p(x)\exp(r_0 x) - r_0 p(x)\exp(r_0 x) \\ &= p'(x)\exp(r_0 x)\end{aligned}$$

By repeatedly applying this differential operator, then the DE states:

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} - r_0\right]^k w(x) = p^{(k)}(x)\exp(r_0 x) = 0.$$

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This implies

$$p^{(k)}(x) = 0 \implies p(x) = c_0 + c_1 x + \cdots + c_{k-1}x^{k-1} = \sum_{j=0}^{k-1} c_j x^j.$$

so that $w(x) = p(x)\exp(r_0 x)$ now identifies k linearly independent functions.

Variation of parameters (3/3)

In summary, if the characteristic equation for a DE has k repeated roots with $r = r_0$, then the k linearly indpendent solutions associated with this root are:

$$\begin{aligned} y_1(x) &= \exp(r_0 x), \\ y_2(x) &= x \exp(r_0 x), \\ y_3(x) &= x^2 \exp(r_0 x), \\ &\cdots \\ y_k(x) &= x^{k-1} \exp(r_0 x). \end{aligned}$$

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Example (Example 5.3.2)

Compute the general solution to the DE

$$9y^{(5)} - 6y^{(r)} + y^{(3)} = 0.$$

Complex valued roots

Recall that if an algebraic equation

$$\sum_{j=0}^{n} a_j r^j = 0,$$

with real-valued coefficients a_j has a complex root, then its complex conjugate is also a root.

I.e., any complex-valued roots r come in *conjugate pairs* for real-valued algebraic equations:

$$r = \sigma \pm i\omega, \qquad\qquad i := \sqrt{-1}.$$

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$$r = \sigma \pm i\omega, \qquad\qquad i := \sqrt{-1}.$$

where σ and ω are real numbers. Before proceeding, we review some basic facts about complex arithmetic: with x and y real numbers:

$$z = x + iy, \qquad\qquad |z| = \sqrt{x^2 + y^2}.$$

Complex valued roots

A fundamental tool is **Euler's identity**:

$$\exp(i\theta) = \cos\theta + i\sin\theta,$$

and so all complex number have a *polar* form:

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Complex-valued roots are not a problem, in principle: if $r = \sigma \pm i\omega$ are conjugate roots of the characteristic equation, then

$$y_1(x) = \exp((\sigma + i\omega)x), \qquad y_2(x) = \exp((\sigma - i\omega)x)$$

are two linearly independent solutions to the DE. (Check their linear independence!)

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Example

Compute the general solution to $y'' + y = 0$.

Trigonometric solutions

While complex-valued solutions to DE's are in principle fine, they are difficult to physically interpret if the DE models a real-valued system.

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Euler's identity can help rewrite complex exponentials in real-valued form: if two characteristic equation roots are $\sigma \pm i\omega$, then a linear combination of them can be rearranged:

$$\begin{aligned} c_1 \exp((\sigma + i\omega)x) + c_2 \exp((\sigma - i\omega)x) &= \\ (c_1 + c_2) \exp(\sigma x) \cos(\omega x) + i(c_1 - c_2) \exp(\sigma x) \sin(\omega x) &= \\ c_3 \exp(\sigma x) \cos(\omega x) + c_4 \exp(\sigma x) \sin(\omega x). \end{aligned}$$

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Thus, the *real* part of the root becomes the exopnential coefficient, and the *imaginary* part becomes the frequency of trigonometric functions.

Roots of the characteristic equation

In summary:

- For real-valued distinct roots r_j, then $y_j = \exp(r_j x)$.
- For a k-fold repeated real-valued root r_0, then $y_1 = \exp(r_0 x), \ldots, y_k = x^{k-1}\exp(r_0 x)$.
- For a complex conjugate root pair $r = \sigma \pm i\omega$, then $y_1(x) = \exp(\sigma x)\cos(\omega x)$ and $y_2(x) = \exp(\sigma x)\sin(\omega x)$.

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Example (Example 5.3.4)

Find the particular solution of $y'' - 4y' + 5y = 0$, for which $y(0) = 1$ and $y'(0) = 5$.

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Example (Example 5.3.5)

Compute the general solution of $y^{(4)} + 4y = 0$.