Basis and dimension

MATH 2250 Lecture 19
Book section 4.4

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Basis for vector spaces

We begin with a foundational definition in linear algebra:

The vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are a **basis** for a vector space V if

- $\mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} = V$
- $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are linearly independent.

Note: basis sets are <u>not</u> unique. There are infinitely many bases for any given V.

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We've already seen the utility of a basis: any element of V has a unique coordinate representation in a basis.

Dimension

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The dimension of $\mathbb{R}^n$ is n.
Most vector spaces in this class have finite dimension.
But there are plenty of important infinite-dimensional spaces.

Linear independence, span, basis, and dimension

To summarize all these terms: let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ be a generic set of vectors in the n-dimensional vector space V. Then:

- If S is comprised of linearly independent vectors, then S is a subset of some basis for V.
- If S spans V, then some subset of S forms a basis for V .
- If S spans V and contains n vectors, then it's a basis for V.
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Example (Example 4.4.5)

Find a basis for the solution space of the homogeneous linear system:

$$\begin{aligned} 3x_1 + 6x_2 - x_3 - 5x_4 + 5x_5 &= 0 \\ 2x_1 + 4x_2 - x_3 - 3x_4 + 2x_5 &= 0 \\ 3x_1 + 6x_2 - 2x_3 - 4x_4 + x_5 &= 0 \end{aligned}$$