Linear independence

MATH 2250 Lecture 18
Book section 4.3

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Linear combinations

We've seen notions of linear independence in $\mathbb{R}^3$ and $\mathbb{R}^n$. This section: we investigate this in more detail.

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If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are a set of given k vectors in in $\mathbb{R}^n$, then another vector $\boldsymbol{w}$ is a **linear combination** of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ if there are constants $c_1, \ldots, c_k$ such that

$$\sum_{j=1}^{k} c_j \boldsymbol{v}_j = c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 + \cdots + c_k \boldsymbol{v}_k = \boldsymbol{w}.$$

Note that no assumptions about the vectors $\boldsymbol{v}_j$ is made.
There is also no prescribed relationship between k and n.

Examples

Example (Example 4.3.1)

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Determine whether or not $\boldsymbol{w} = (-7, 7, 11)$ in $\mathbb{R}^3$ is a linear combination of $\boldsymbol{v}_1 = (1, 2, 1)$, $\boldsymbol{v}_2 = (-4, -1, 2)$, and $\boldsymbol{v}_3 = (-3, 1, 3)$.

Spanning sets

Given a vector space V (say $V = \mathbb{R}^n$), we are interested in determining when a set of vectors "represents" the vector space V.

We say that the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ **span** the vector space V if every vector in V is a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$.

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Example

In $\mathbb{R}^n$, the canonical example of a spanning set are the cardinal unit vectors:

$$\begin{aligned} \boldsymbol{e}_1 &= (1, 0, 0, \cdots 0, 0) \\ \boldsymbol{e}_2 &= (0, 1, 0, \cdots 0, 0) \\ &\vdots \\ \boldsymbol{e}_n &= (0, 0, 0, \cdots 0, 1). \end{aligned}$$

The vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ span $\mathbb{R}^n$ since every vector in $\mathbb{R}^n$ is a linear combination of these.

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Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ be the set of these vectors.

The following are all equivalent notation/terminology:

- $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ span W
- S is a spanning set for W
- $\mathrm{span}(S) = W$
- $\mathrm{span}\{\boldsymbol{v}_1, \ldots \boldsymbol{v}_k\} = W$.

Spanning sets and linear independence

If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ spans W, then every vector in W is expressible in coordinates of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$.

If $\boldsymbol{w} \in W$, by definition there are constants $c_1, \ldots, c_k$ such that

$$\boldsymbol{w} = \sum_{j=1}^{k} c_j \boldsymbol{v}_j.$$

The vector $(c_1, \ldots, c_k)$ are the **coordinates** or the **coordinate representation** of $\boldsymbol{w}$ in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$.

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Recall that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are *linearly independent* if the equation

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The coordinate representation of elements in W by the spanning set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is unique if and only if $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are linearly independent.

Spanning sets and linear independence in $\mathbb{R}^n$

In $\mathbb{R}^n$, we have a simple way to test if a set of vectors is linearly independent:

Theorem

The vectors $\boldsymbol{v}_1, \ldots \boldsymbol{v}_n$ *in* $\mathbb{R}^n$ *are linearly independent if and only if* $\det(\boldsymbol{A}) \neq 0$, *where* $\boldsymbol{A}$ *is the* $n \times n$ *matrix:*

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$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{pmatrix}$$

Note that *more* than n vectors in $\mathbb{R}^n$ must be linearly dependent. (The reduced Echelon form for the associated $\boldsymbol{A}$ is not the identity.)

Fewer than n vectors in $\mathbb{R}^n$ may or may not be linearly independent.