The vector space $\mathbb{R}^n$

MATH 2250 Lecture 17
Book section 4.2

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Vectors in $\mathbb{R}^n$

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And because of the behavior of n-tuple sums and multiplication by scalars, then the set of all n-tuples, $\mathbb{R}^n$, is a vector space.

$\mathbb{R}^n$ is a vector space

The set of points $\mathbb{R}^n$ is a vector space:

If $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are vectors in $\mathbb{R}^n$ and r and s are any scalars, then:

- $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
- $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$
- $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{u} = \boldsymbol{u}$ ($\boldsymbol{0}$ a vector with all entries 0)
- $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$
- $r(\boldsymbol{u} + \boldsymbol{v}) = r\boldsymbol{u} + r\boldsymbol{v}$
- $(r + s)\boldsymbol{u} = r\boldsymbol{u} + s\boldsymbol{u}$
- $r(s\boldsymbol{u}) = (rs)\boldsymbol{u}$
- $1\,\boldsymbol{u} = \boldsymbol{u}$

The properties above are also practically and mathematically important.

Subspaces

Since $\mathbb{R}^n$ is a vector space, we have the concept of *subspaces*.

Given a vector space V, a **subspace** is any subset of V satisfying the definition of a vector space.

I.e., is a subset such that (a) adding two elements in the subspace together results in something in the same subspace and (b) multiplying any element by a scalar results in a vector again in the subspace.

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If W is a subspace of V, and $\boldsymbol{u}$ and $\boldsymbol{v}$ are any elements in W, then

- $\boldsymbol{u} + \boldsymbol{v}$ is also in W
- $c\boldsymbol{u}$ is also in W for any scalar c.

Subspaces examples

Example

In $\mathbb{R}^4$, is the set of tuples (x_1, x_2, x_3, x_4) satisfying **both**

$$\begin{aligned} x_1 + 3x_3 + 4x_4 &= 0 \\ -x_1 - 2x_2 + x_3 + x_4 &= 0 \end{aligned}$$

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Example

Let $\boldsymbol{A}$ be an $m \times n$ matrix. Let V be the set of vectors $\boldsymbol{x}$ in $\mathbb{R}^n$ satisfying

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}.$$

Is V a subspace?

Subspaces examples, part 2

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Let V be the set of vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ of $\mathbb{R}^3$ such that $x_1 + 2x_2 - 3x_3 = 1$. Is V a subspace?

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Let V be the set of functions $f(\cdot)$ that map $\mathbb{R}$ to $\mathbb{R}$. Let W be the set of functions in V that are polynomials of degree 3 or less. Is W a subspace of V?