Matrix determinants

MATH 2250 Lecture 15
Book section 3.6

September 23, 2019

2×2 determinants

Given a 2×2 matrix,

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the **determinant** of $\boldsymbol{A}$ is denoted and defined as

$$\det(\boldsymbol{A}) = |\boldsymbol{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Note that the determinant of a matrix is a scalar, and that we have several ways to denote a determinant.

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Recall that the factor $ad - bc$ appeared in our explicit formula for the inverse of 2×2 matrices – this is not coincidental.

Determinants play an essential role in linear algebra and linear systems.

Computing higher-order determinants

To compute the determinant of a 3×3 matrix, the following formula can be used:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The pattern is: we "expand" along the first row, and the 2×2 determinants multiplying the numbers in the first row are formed from matrices where we eliminate elements in the same row and column as the numbers.

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Example

Compute the following determinants:

$$\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 4 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{vmatrix},$$

A general procedure: minors and cofactors

Let $\boldsymbol{A}$ be an $n \times n$ matrix.

The i, j **minor** of $\boldsymbol{A}$ is $M_{i,j}$, the *determinant* of the $(n-1) \times (n-1)$ matrix formed by deleting row i and column j from $\boldsymbol{A}$.

The i, j **cofactor** of $\boldsymbol{A}$ is $(-1)^{i+j} M_{i,j}$.

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To compute a general determinant, one can *expand in cofactors along row* i,

$$\det(\boldsymbol{A}) = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} M_{i,j}$$

for any $i = 1, \cdots, n$. Or we can *expand in cofactors along column* j

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Note: one can choose *any* row or column to expand along; the answer will be the same.

One typically takes advantages of this and expands along the row/column with the maximum number of zeros.

Examples

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Compute the determinant

$$\begin{vmatrix} 7 & 6 & 0 \\ 9 & -3 & 2 \\ 4 & 5 & 0 \end{vmatrix}.$$

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Example (Exercise 3.6.12)

Compute the determinant

$$\begin{vmatrix} 2 & 0 & 0 & -3 \\ 0 & 1 & 11 & 12 \\ 0 & 0 & 5 & 13 \\ -4 & 0 & 0 & 7 \end{vmatrix}.$$

Properties of determinants

Let $\boldsymbol{A}$ be an $n \times n$ matrix. The following are properties of the determinant:

- For a constant c, then $\det(c\boldsymbol{A}) = c^n \det \boldsymbol{A}$
- If a matrix $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by interchanging two rows of $\boldsymbol{A}$, then $\det \boldsymbol{B} = -\det \boldsymbol{A}$
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There are far more interesting properties of determinants, some of which we'll use later in this class:

- $\det(\boldsymbol{A}\boldsymbol{B}) = \det \boldsymbol{A} \det \boldsymbol{B}$
- $\boldsymbol{A}$ is singular (i.e., non-invertible) if and only if $\det \boldsymbol{A} = 0$.
- The (i, j) entry of $\boldsymbol{A}^{-1}$ is $(-1)^{i+j} M_{j,i}/|\boldsymbol{A}|$