Matrix inverses

MATH 2250 Lecture 14
Book section 3.5

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The identity matrix

We first define a special matrix, the $n \times n$ **identity** matrix,

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

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This matrix effects multiplicative identity: if $\boldsymbol{A}$ is a matrix comprised of columns $\boldsymbol{a}_j$

$$\boldsymbol{A} = [\boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \cdots \;\; \boldsymbol{a}_n],$$

then via matrix multiplication operations, we can see that

$$\boldsymbol{A}\boldsymbol{I} = [\boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \cdots \;\; \boldsymbol{a}_n] = \boldsymbol{A}.$$

Similarly, one can show that $\boldsymbol{I}\boldsymbol{B} = \boldsymbol{B}$ for any matrix $\boldsymbol{B}$ with n rows.

Matrix inverses

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Let $\boldsymbol{A}$ be a square $n \times n$ matrix. We seek to compute a new $n \times n$ matrix $\boldsymbol{B}$ such that

$$\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A} = \boldsymbol{I}.$$

If we can find such a matrix, we call $\boldsymbol{B}$ the **matrix inverse** of $\boldsymbol{A}$, and write $\boldsymbol{B} = \boldsymbol{A}^{-1}$.

If $\boldsymbol{A}$ has a matrix inverse, we say that $\boldsymbol{A}$ is **invertible**.

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Example

Determine whether or not the following matrix is invertible, and if so compute its inverse.

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

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But with matrices, even if $\boldsymbol{A}$ is not the zero matrix, sometimes we cannot invert it.

However, if an inverse exists, then it's unique:

Theorem

If an $n \times n$ matrix $\boldsymbol{A}$ has a matrix inverse $\boldsymbol{A}^{-1}$, then $\boldsymbol{A}^{-1}$ is the unique matrix inverse of $\boldsymbol{A}$.

2×2 matrix inverses

There is an explicit formula for the inverse of a 2×2 matrix:

Theorem

Consider a general 2×2 *matrix:*

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $\boldsymbol{A}$ *is invertible if and only if* $ad - bc \neq 0$, *and in this case its inverse is*

$$\boldsymbol{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

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For completeness, we note that if $\boldsymbol{A}$ is invertible, then

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Inverses of matrix products involves a reversal of order: if $\boldsymbol{A}$ and $\boldsymbol{B}$ are both invertible matrices of the same size, then

$$(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}.$$

Furthermore, for any integer $k \geqslant 1$,

$$\left(\boldsymbol{A}^{-1}\right)^{k} = \left(\boldsymbol{A}^{k}\right)^{-1}$$

Linear systems and matrix inverses

Recall that we often write linear systems with n equations and n unknowns in matrix form as

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},$$

where $\boldsymbol{A}$ is a known $n \times n$ matrix, $\boldsymbol{b}$ is a known $n \times 1$ vector, and $\boldsymbol{x}$ is the unknown $n \times 1$ vector.

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Example

Use matrix inverses to solve the system

$$\begin{aligned} 2x_1 + x_2 &= 3 \\ -x_1 + x_2 &= 6 \end{aligned}$$

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Example

Use Gaussian elimination to compute the matrix inverse of

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

3×3 matrix inverse

Example

Compute the inverse of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{pmatrix}$$

Singular matrices

Matrix invertibility has strong connections to solutions of linear systems.

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Theorem

If $\boldsymbol{A}$ is a square $n \times n$ matrix, the following are all equivalent statements:

1. *$\boldsymbol{A}$ is invertible.*
2. *$\boldsymbol{A}$ is non-singular (i.e., $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ is true only for $\boldsymbol{x} = \mathbf{0}$)*
3. *$\boldsymbol{A}$ is row-equivalent to $\boldsymbol{I}$.*
4. *The equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for any $n \times 1$ vector $\boldsymbol{b}$.*