Matrix operations

MATH 2250 Lecture 13
Book section 3.4

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Manipulations of matrices

If x and y are numbers, we have standard definitions for operations like

$$x + y, \quad x - y, \quad xy, \quad \frac{x}{y}.$$

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If $\boldsymbol{A}$ is an $m \times n$ matrix with $m = n = 1$, then $\boldsymbol{A} = A$ is called a *scalar*. Hence, we have arithmetic rules for scalars.

We now explore similar notions for matrices.

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Multiplication of a matrix and a scalar: Let $\boldsymbol{A}$ be an $m \times n$ matrix, and c be a scalar.

$$c\boldsymbol{A},$$

is an elementwise operations, multiplying each element by c, whose output is another $m \times n$ matrix.

Example

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For arbitrary scalars s and t, we have

$$\begin{pmatrix} 3-4s \\ t+s \\ -t-4 \\ -1+s+t \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -4 \\ -1 \end{pmatrix} + s \begin{pmatrix} -4 \\ 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

Elementwise addition

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are of different sizes, then $\boldsymbol{A} \pm \boldsymbol{B}$ is undefined.

Example

If

$$\boldsymbol{x} = \begin{pmatrix} 3 & 2 & -1 \end{pmatrix}, \qquad \boldsymbol{y} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix},$$

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If $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are of the same size, then addition is commutative and associative:

$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}, \qquad \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}.$$

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So far, this is all just the same as arithmetic for scalars.

Matrix multiplication

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A simple example of the type of multiplication that we need:

$$\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \implies \boldsymbol{x}\boldsymbol{y} = \sum_{j=1}^{n} x_j y_j.$$

Thus, $\boldsymbol{x}$ is $1 \times n$, and $\boldsymbol{y}$ is $n \times 1$.
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Note that the product of these two vectors is formed by *summing elementwise products*.

We will use this general principle to define multiplication of general matrices.

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I.e., $\boldsymbol{AB}$ makes sense if and only if $\boldsymbol{A}$ is $m \times p$ and $\boldsymbol{B}$ is $p \times n$, where n, p, and m are arbitrary.

- The output matrix $\boldsymbol{C} = \boldsymbol{AB}$ has size $m \times n$.
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Example

Compute the following matrix products:

$$\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 5 & 3 \\ -4 & 2 \end{pmatrix}$$

Matrix multiplication example

Example (Example 3.4.7)

Write the following system of linear equations as a vector equality involving a matrix-vector product:

$$\begin{aligned} 3x_1 - 4x_2 + x_3 + 7x_4 &= 10 \\ 4x_1 - 5x_3 + 2x_4 &= 0 \\ x_1 + 9x_2 + 2x_3 - 6x_4 &= 5. \end{aligned}$$

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There are even stranger things that happen with matrix multiplication:

Example (Example 3.4.8)

Let

$$\boldsymbol{A} = \begin{pmatrix} 4 & 1 & -2 & 7 \\ 3 & 1 & -1 & 5 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 & 5 \\ 3 & -1 \\ -2 & 4 \\ 2 & -3 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} 3 & 4 \\ 2 & 1 \\ -2 & 3 \\ 1 & -3 \end{pmatrix}.$$

Show that $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{A}\boldsymbol{C}$, even though $\boldsymbol{B} \neq \boldsymbol{C}$.