Gaussian Elimination

MATH 2250 Lecture 11
Book section 3.2

September 16, 2019

Abbreviated linear systems

From the previous lecture, we wanted to solve the system

$$
\begin{aligned}
1x + 2y + 1z &= 4 \\
3x + 8y + 7z &= 20 \\
2x + 7y + 9z &= 23.
\end{aligned}
$$

Abbreviated linear systems

From the previous lecture, we wanted to solve the system

$$\begin{aligned} 1x + 2y + 1z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 9z &= 23. \end{aligned}$$

Our procedure to solve this system did not require us to continually write the (x, y, z) variables.

Thus, an abbreviated version of the above system could be:

$$\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}$$

Such a tabular arrangement of numbers is a **matrix**.

Matrices

Matrices come in all sizes and shapes. We say a matrix is $m \times n$ if it has m rows and n columns.

$$\begin{bmatrix} 1 & 3 & -2 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 0 & \pi \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ -3 & 7 \\ 16 & 0 \end{bmatrix}$$

These matrices are 1×3, 2×2, and 4×2, respectively.

Matrices

Matrices come in all sizes and shapes. We say a matrix is $m \times n$ if it has m rows and n columns.

$$\begin{bmatrix} 1 & 3 & -2 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 0 & \pi \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ -3 & 7 \\ 16 & 0 \end{bmatrix}$$

These matrices are 1×3, 2×2, and 4×2, respectively.

Some terminology: if $m = n$, the matrix is **square**. Otherwise it is **rectangular**.
The individual numbers in matrices are **elements**.
Matrices are also called **arrays**.

Matrices

Matrices where m or n equals 1 are also called **vectors**. The matrices

$$\left[\begin{array}{ccc} 1 & -3 & 2 \end{array}\right], \qquad \left[\begin{array}{c} 1 \\ -3 \end{array}\right],$$

are a row and column vector, respectively.

Matrices

Matrices where m or n equals 1 are also called **vectors**. The matrices

$$\begin{bmatrix} 1 & -3 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ -3 \end{bmatrix},$$

are a row and column vector, respectively.

We will often denote vectors as lowercase bold letters, and matrices as uppercase bold letters, e.g.,

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \boldsymbol{A} = \begin{bmatrix} 1 & -3 & 4 \\ 2 & 0 & 0 \end{bmatrix}$$

Matrices

Matrices where m or n equals 1 are also called **vectors**. The matrices

$$\begin{bmatrix} 1 & -3 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ -3 \end{bmatrix},$$

are a row and column vector, respectively.

We will often denote vectors as lowercase bold letters, and matrices as uppercase bold letters, e.g.,

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \qquad \boldsymbol{A} = \begin{bmatrix} 1 & -3 & 4 \\ 2 & 0 & 0 \end{bmatrix}$$

If $\boldsymbol{A}$ is an $m \times n$ matrix, the element in the ith row and jth column is $A(i,j)$.

Linear systems as matrices

We will often use the correspondence between linear systems and matrices. For the system,

$$
\begin{aligned}
1x + 2y + 1z &= 4\\
3x + 8y + 7z &= 20\\
2x + 7y + 9z &= 23,
\end{aligned}
$$

the following two matrices can be associated with this system:

$$
\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1\\ 3 & 8 & 7\\ 2 & 7 & 9 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 4\\ 20\\ 23 \end{pmatrix}.
$$

Linear systems as matrices

We will often use the correspondence between linear systems and matrices. For the system,

$$\begin{aligned} 1x + 2y + 1z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 9z &= 23, \end{aligned}$$

the following two matrices can be associated with this system:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 8 & 7 \\ 2 & 7 & 9 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 4 \\ 20 \\ 23 \end{pmatrix}.$$

- $\boldsymbol{A}$ is called the **coefficient** matrix for this system
- The 3×4 **concatenated** matrix $[\boldsymbol{A} \;\; \boldsymbol{b}]$ is called the **augmented coefficient matrix**.

Row operations on matrices

We solved linear systems by manipulating and combining the equations.
For matrix representations, this translates to **row operations** on matrices.

Row operations on matrices

We solved linear systems by manipulating and combining the equations. For matrix representations, this translates to **row operations** on matrices.

- Multiply row j by a constant c (multiply equation j by c)
- Swap/interchange row j with row k (interchange equation j and equation k)
- Replace row j with row j plus row k (replace equation j with equation j plus equation k)

Two matrices that differ from each other by only these row operations are called **row equivalent**.

Row operations on matrices

We solved linear systems by manipulating and combining the equations.
For matrix representations, this translates to **row operations** on matrices.

- Multiply row j by a constant c (multiply equation j by c)
- Swap/interchange row j with row k (interchange equation j and equation k)
- Replace row j with row j plus row k (replace equation j with equation j plus equation k)

Two matrices that differ from each other by only these row operations are called **row equivalent**.

Example

Use row operations to solve the linear system

$$\begin{aligned} 1x + 2y + 1z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 9z &= 23, \end{aligned}$$

Gaussian elimination

The formulaic procedure we use to solve linear systems via row operations is called **Gaussian elimination**.

The steps are:

- Reduce the augmented matrix to **Echelon form**
  - Use row operations to eliminate leading variables
  - Swap rows as necessary so that the system is “upper triangular"
  - A matrix in this upper triangular form is said to be an Echelon matrix
- Use **back substitution** to solve for variables
  - Solve the last equation first
  - Sequentially work up the rows of the matrix
  - Set “free" variables equal to a parameter; solve for the other variables in terms of the free variables.

Examples

Example (Example 3.2.4)

Use back substitution to solve the following equations (whose augmented matrix is already in Echelon form).

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ x_3 + 2x_5 &= -3 \\ x_4 - 4x_5 &= 7 \end{aligned}$$

Examples

Example (Example 3.2.4)

Use back substitution to solve the following equations (whose augmented matrix is already in Echelon form).

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ x_3 + 2x_5 &= -3 \\ x_4 - 4x_5 &= 7 \end{aligned}$$

Example (Example 3.2.5)

Use Gaussian elimination to solve the following system of equations.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ 2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 &= 7 \\ 3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 &= 27 \end{aligned}$$