Introduction to linear systems

MATH 2250 Lecture 10
Book section 3.1

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Linear equations

The subject of linear algebra studies linear equations. A **linear equation** in n variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \sum_{j=1}^{n} a_j x_j = b,$$

for some constants $a_1, a_2, \ldots, a_n$ and b.

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Nonlinear equations are, for example,

$$x_1^2 + x_2 = 3, \qquad x_1 x_2 + x_3 = 9, \qquad x_1 + \exp(x_2) = 0,$$

and are not studied here.

Linear systems

A **linear system** is a collection of 1 or more linear equations, e.g.,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

for some constants $a_{11}, a_{12}, \ldots, a_{mn}$ and $b_1, \ldots, b_m$.

These equations, collectively, are a set of constraints (m constraints) on a set of variables (n variables).

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A set of values $(x_1, \ldots, x_n)$ that satisfies all m constraints simultaneously is a "solution".

The number of constraints m need not be related to the number of variables n.

Solutions to linear systems need not exist, and they need not be unique.

Examples: 2 variables, 2 constraints

We can gain understanding about the general case by considering examples in the "2×2" case.

Example

Compute all solutions to the linear system

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_1 - 2x_2 &= 0 \end{aligned}$$

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Example

Compute all solutions to the linear system

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_1 - 2x_2 &= 0 \end{aligned}$$

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$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_1 + 2x_2 &= 0 \end{aligned}$$

Examples: 2 variables, 2 constraints

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Compute all solutions to the linear system

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 2x_1 + 4x_2 &= 8 \end{aligned}$$

Examples: 2 variables, 2 constraints

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Compute all solutions to the linear system

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 2x_1 + 4x_2 &= 8 \end{aligned}$$

In the general case of m equations with n unknowns, it turns out linear systems can only have 1, 0, or infinitely many solutions.

I.e., the 3 examples above are comprehensive exemplars of the number of solutions to linear systems.

Examples: 3 variables, 3 constraints

Example (Example 3.1.6)

Compute all solutions to the linear system

$$\begin{aligned} x + 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 9z &= 23 \end{aligned}$$

Example (Example 3.1.7)

Compute all solutions to the linear system

$$\begin{aligned} 3x - 8y + 10z &= 22 \\ x - 3y + 2z &= 5 \\ 2x - 9y - 8z &= -11 \end{aligned}$$

Linear systems and DE's

Linear systems appear everywhere in science and engineering.
In this class we will mostly use linear systems for modest goals, such as the following.

Example (Example 3.1.8)

Verify that $y(x) = A\exp(3x) + B\exp(-3x)$ is a solution to the DE

$$y'' - 9y = 0,$$

for arbitrary values of the constants A and B. Find the particular values of (A, B) that that y satisfies the initial data

$$y(0) = 7, \qquad\qquad y'(0) = 9.$$