First-order linear differential equations

MATH 2250 Lecture 05
Book section 1.5

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Linear equations

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$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y),$$

is too difficult for us to solve in general.

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However, a special class of first-order equations can always be solved explicitly.

A first-order equation is a **linear** DE if it has the form,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x),$$

for some functions P and Q

If P and Q are continuous, we can explicitly solve for unique solutions.

The "trick"

We can always solve first-order linear DE's explicitly.

The idea revolves around generalizing the following example:

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But notice that if we multiply by x^2:

$$\begin{aligned} x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy &= x^2 Q(x), \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 y\right) &= x^2 Q(x), \end{aligned}$$

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We can now simply integrate both sides of the equation wrt x:

$$\begin{aligned} \int \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 y\right)\mathrm{d}x &= \int x^2 Q(x)\mathrm{d}x, \\ x^2 y(x) &= \int x^2 Q(x)\mathrm{d}x. \end{aligned}$$

Assuming we can evaluate the right-hand side integral, we can then easily solve for $y(x)$.

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It turns out that the "correct" multiplication factor is $\rho(x)$, given by

$$\rho(x) = \exp\left(\int P(x)\mathrm{d}x\right)$$

Note: The constant of integration after computing $\int P(x)\mathrm{d}x$ can be ignored (and hence set to 0). (Why?)

The function $\rho(x)$ above is called an **integrating factor**.

Solutions to first-order linear DE's

Our DE is

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Thus, the full solution is

$$y(x) = \frac{C}{\rho(x)} + \frac{1}{\rho(x)}\int \rho(x)Q(x)\mathrm{d}x,$$

where C is determined from initial data.

Note: This formula should **not** be memorized.

An "algorithm" for solving first-order linear DE's

The overall procedure for solving a first-order linear DE is as follows:

1. Ensure the DE is first-order linear. Write in standard form.
2. Compute the integrating factor $\rho(x)$
3. Multiply both sides by $\rho(x)$.
4. If the previous steps are done correctly, the left-hand side can and should be written as the derivative of a product.
5. Integrate both sides wrt x, and solve for $y(x)$.
6. Apply available initial data to determine the unknown constant.

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$$(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} + 3xy = 6x, \qquad\qquad y(1) = 4$$

Existence and uniqueness

Linear equations are very convenient: we have existence and uniqueness theory that is stronger than the general case:

Theorem

Given a first-order linear IVP in standard form:

$$y' + P(x)y = Q(x), \qquad\qquad y(x_0) = y_0,$$

assume that P and Q are both continuous in an interval I containing x_0. Then there is a unique solution to the IVP on the interval I.