Slope fields and solution curves

MATH 2250 Lecture 03
Book section 1.3

August 23, 2019

First-order DE's

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The good news: we can gain understanding of the *character* of the solutions fairly easily.

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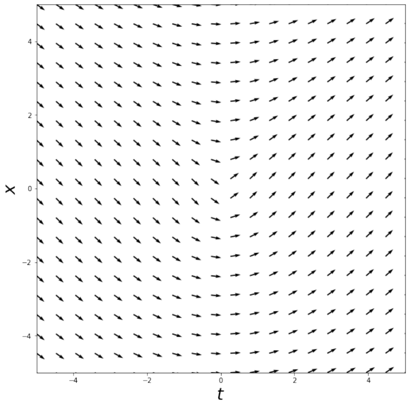
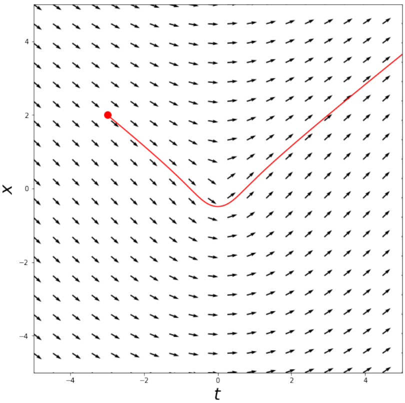
Draw slope field for

$$y' = x - y$$

Solution curves

Slope fields give us a way to approximate solution curves:





Demo

See `slopefield.ipynb`

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Both of the situations above cause practical and philosophical problems.

Existence and uniqueness guarantees

Theorem

For the initial value problem,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y), \qquad\qquad y(x_0) = y_0,$$

if both $f(x,y)$ *and* $f_y(x,y) = \frac{\partial}{\partial y} f(x,y)$ *are continuous in an* (x,y) *region containing* (x_0, y_0)*, then this problem has a unique solution on an interval* I *containing* x_0*.*

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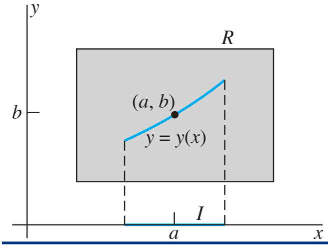

In practice we can easily verify that the right-hand side function and its y-derivative are continuous near the initial data.

The interval I where a unique solution exists can be very large or very small – this theorem does not give insight into this.

Existence and uniqueness examples

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Existence and uniqueness examples

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Does the initial value problem

$$y' = \frac{x}{x^2 + y^2}, \qquad y(-1) = 3,$$

have a unique solution in a neighborhood around $x = -1$?