2250 - Practice midterm 1

1.) (a) $y(x) = \sqrt{\frac{x^3+C}{x}}$, C arbitrary

DE: $y' = \frac{3x^2-y^2}{2xy} \longrightarrow 2xyy' = 3x^2 - y^2$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{x^3+C}{x}\right)^{-1/2} \cdot \left(2x - \frac{C}{x^2}\right) = \frac{1}{2y}\left(2x - \frac{C}{x^2}\right)$$

$$\text{LHS} = 2xyy' = 2xy \cdot \frac{1}{2y}\left(2x - \frac{C}{x^2}\right)$$

$$= 2x^2 - \frac{C}{x}$$

$$\text{RHS} = 3x^2 - y^2 = 3x^2 - \frac{x^3+C}{x}$$

$$= 2x^2 - \frac{C}{x}$$

equal ✓.

(b) $y = \sqrt{\frac{x^3+C}{x}}$, $y(0) = 3 \Rightarrow 3 = \sqrt{\frac{0+C}{0}} \longrightarrow$ no solution

(typo in document, so replace with, e.g. $y(1)=3$)

$$y(1) = 3 \Rightarrow 3 = \sqrt{\frac{1+C}{1}}$$

$$9 = 1 + C \Rightarrow C = 8$$

$$y(x) = \sqrt{\frac{x^3+8}{x}}$$

2.) (a) $y' = t\exp(-t)$ $y(0) = \pi$

$$\int \frac{dy}{dt}\,dt = \int t e^{-t}\,dt$$

integration by parts

$u = t$ $v = -e^{-t}$

$u' = 1$ $v' = e^{-t}$

$$= uv - \int v \cdot u'\,dt = -te^{-t} + \int e^{-t}\,dt$$

$$= -te^{-t} - e^{-t} + C.$$

$$y(t) = -e^{-t}(1+t) + C.$$

$y(0) = \pi \Rightarrow \pi = -1(1+0) + C$

$C = 1 + \pi$

$$y(t) = -e^{-t}(1+t) + 1 + \pi$$

(b) $y' + 4y = t$, $y(0) = 1$

integrating factor: $\rho(t) = \exp(\int 4\,dt) = e^{4t}$

$$e^{4t}y' + 4e^{4t}y = te^{4t}$$

$$\frac{d}{dt}(ye^{4t}) = te^{4t}$$

$$ye^{4t} = \int te^{4t}\,dt \xrightarrow{\text{IBP}} ye^{4t} = \frac{1}{4}te^{4t} - \frac{1}{4}\int e^{4t}\,dt$$

$u = t$ $v = \frac{1}{4}e^{4t}$

$u' = 1$ $v' = e^{4t}$

$$= \frac{1}{4}e^{4t}\left(t - \frac{1}{4}\right) + C.$$

$y(t) = \frac{1}{4}\left(t - \frac{1}{4}\right) + Ce^{-4t}$

$y(0) = 1 \Rightarrow 1 = -\frac{1}{16} + C, \quad C = 17/16$

$y(t) = \frac{1}{4}\left(t - \frac{1}{4}\right) + 17/16\, e^{-4t}$

3.). (a) $x' = k(x - x^2)$

Critical points: $k \neq 0$: $k(x - x^2) = 0 \rightarrow x = 0, 1$.

$k = 0$: $0 = 0 \Rightarrow$ all x are equilibrium solutions

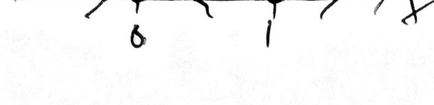
(b). $k=0$: all CP's are unstable since perturbing initial data does not ensure solution returns to unperturbed state

(This is a pathological case that I don't really expect Students to study in depth. Such a case won't be on the exam)

$k \neq 0$: $k > 0$: phase diagram 

$x=0$ unstable

$x=1$ Stable

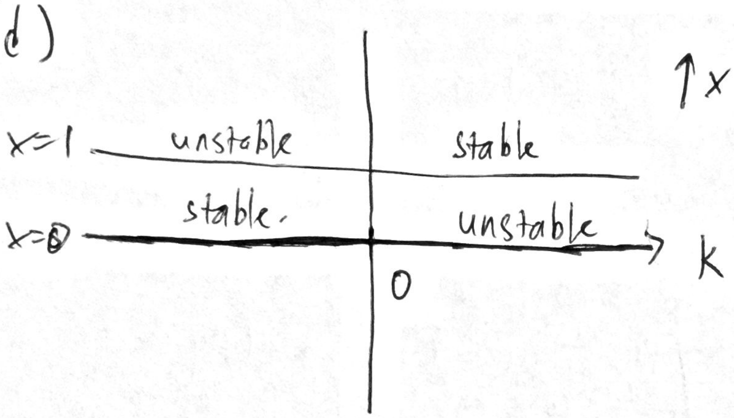
$k < 0$: phase diagram 

$x=0$: stable

$x=1$: unstable.

(c). Stability changes at $k=0 \Rightarrow k=0$ is a bifurcation point

(d)



4.) (a) T: temperature (celsius)

$$\frac{dT}{dt} = k(25-T). \quad k\text{: unknown constant}, \ k>0.$$

(b) Separate variables:

$$\int \frac{dT}{25-T} = \int k\, dt$$

$$-\log|25-T| = kt + C$$

$$25-T = C\exp(-kt) \quad (C \leftarrow 1/\exp(c))$$

$$T(t) = 25 - C\exp(-kt), \quad C \text{ arbitrary}$$

(c) $T(0)=50 \Rightarrow 50 = 25 - C$

$$C = -25$$

$$T(t) = 25 + 25\exp(-kt).$$

(d) $T(0)=0, \quad T(5)=20$

$$\Downarrow$$

$$0 = 25 - C \rightarrow C = 25$$

$$T(t) = 25 - 25\exp(-kt)$$

$$T(5)=20 \Rightarrow 20 = 25 - 25\exp(-k\cdot 20)$$

$$1/5 = \exp(-20k) \rightarrow k = -\frac{1}{20}\log(1/5) = \underline{\frac{1}{20}\log 5 = k}$$