Math 2250 Lab 8

31 October 2019

Name: ____________________

uNID: ____________________

Instructions and Due Date:

- **Due:** 7 November 2019 at the start of the lab
- Work together!
- My current office hours: Mon 9:30am-10:30am WEB 1705 Tues 9am-10am LCB basement Math Center, 10am-11am WEB 1705,
- Additional help for the lab: Lab TA's will be in WEB 1705 M, T, W, Th from 8:30am-1pm. If you go there outside of my office hours, you can ask another section's TA for help. Otherwise, you can ask for help from tutors in the Math Center in the LCB basement.

1. **Nonhomogeneous Matrix Equations (Review)**

Consider the following nonhomogeneous matrix equation, where $\mathbf{A}$ is a square matrix

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{1}$$

and its associated homogeneous version

$$\mathbf{A}\mathbf{x} = \mathbf{0}. \tag{2}$$

When $\mathbf{A}$ is invertible both of these equations will have a unique solution. When $\mathbf{A}$ is not invertible, equation (1) will have either no solution or infinitely many solutions, whereas equation (2) will necessarily have infinitely many solutions becasue the trivial solution $\mathbf{x} = \mathbf{0}$ is a solution and thus it is impossible to have no solution.

Suppose $\mathbf{A}$ is not invertible, but $\mathbf{x}_p$ is a solution to (1), and the set $\mathcal{B} = \mathbf{x_1}, \mathbf{x_2}$ is a basis for the solution space of (2). Then

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \mathbf{x}_p \tag{3}$$

is the <u>general solution</u> to (1) for any scalars c_1 and c_2 because

$$\begin{aligned}
\mathbf{A}\mathbf{x} &= A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \mathbf{x}_p) \\
&= c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \mathbf{A}\mathbf{x}_p \\
&= c_1\mathbf{0} + c_2\mathbf{0} + \mathbf{b} \\
&= \mathbf{b}.
\end{aligned}$$

The <u>general solution</u> is an expression which represents <u>all</u> solutions.

Use the above technique to find the <u>general solution</u> to the following nonhomogeneous matrix equation

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -16 \\ 8 \end{bmatrix}. \tag{4}$$

2. **Nonhomogeneous Differential Equations**

   Factorization method:

   Let

$$L = \left[a_2D^2 + a_1D + a_0\right]$$

be a polynomial differential operator (see page 341 of your text), where D represents differentiation with respect to x, that is $D = \dfrac{d}{dx}$, $D^2 = \dfrac{d^2}{dx^2}$ and a_2, a_1, a_0 are scalars. Thus

$$\begin{aligned} Ly &= \left[a_2D^2 + a_1D + a_0\right] y \\ &= a_2D^2y + a_1Dy + a_0y \\ &= a_2y'' + a_1y' + a_0y \end{aligned}$$

Since differentiation is a linear map, L is also a linear map. In other words, it distributes over linear combinations like so,

$$L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2.$$

Consider the following nonhomogeneous differential equation

$$Ly = f(x), \tag{5}$$

and its associated homogeneous version

$$Ly = 0. \tag{6}$$

Suppose y_p is a solution to (5), and the set $\mathcal{F} = \{y_1, y_2\}$ is a basis for the solution space of (6). Then

$$y = c_1y_1 + c_2y_2 + y_p$$

is the general solution to (5) for any scalars c_1 and c_2 because

$$\begin{aligned} Ly &= L(c_1y_1 + c_2y_2 + y_p) \\ &= c_1Ly_1 + c_2Ly_2 + Ly_p \\ &= c_10 + c_20 + f(x) \\ &= f(x) \end{aligned}$$

The general solution is an expression which represents all solutions.

To find a basis for the solution space of equation (6), simply factor L into linear factors, like

$$Ly = (D - a)(D - b)y = 0.$$

Then the basis functions are $y_1 = e^{ax}$ and $y_2 = e^{bx}$ because

$$(D - a)e^{ax} = 0 \qquad \text{and} \qquad (D - b)e^{bx} = 0.$$

Given that $y_p = A\cos(x) + B\sin(x)$ is a solution to

$$\left[D^2 + 5D + 6\right] y = 5\cos(x), \tag{7}$$

for some constants A and B, find the general solution to (7) by doing the following:

(a) Find a basis for the solution space of the associated homogeneous equation for (7) using the factorization method described below.

(b) Find A and B by plugging y_p into equation (7). By equating the coefficients of like trigonometric functions on each side of the resulting equation you will generate a system of two linear equations in the unknowns A and B. Solve that system.

(c) Finally write your general solution as

$$y = c_1 y_1 + c_2 y_2 + y_p.$$

3. (**From basis to differential equation**) In each of the following problems you are given a set of basis functions for the general solution to a homogeneous, linear DE with constant coefficients. Work backwards to determine the DE (in unknown function $y = y(x)$) which they solve.

   Note that:

   1. The number of functions in the basis corresponds to the order of the differential equation.
   2. You can determine all characteristic equation roots and their multiplicities from the basis functions.
   3. From the characteristic equation, you can write the DE.

   For example, if the solutions are e^x and e^{2x}, then the characteristic equation is $(r-1)(r-2) = 0$. Then the corresponding DE is
   $$y'' - 3y' + 2y = 0.$$

   (a) $\{e^{4x}, xe^{4x}, x^2e^{4x}\}$

   (b) $\{e^x, e^{-x}\cos(\sqrt{2}x), e^{-x}\sin(\sqrt{2}x)\}$

(c) $\{1, x, x^2, x^3\}$

(d) $\{e^{-2x}\cos(x), e^{-2x}\sin(x), xe^{-2x}\cos(x), xe^{-2x}\sin(x)\}$