Math 2250 Lab 7

24 Oct 2019

Name: ____________________

uNID: ____________________

Instructions and Due Date:

- **Due:** 31 Oct 2019 at the start of the lab
- Please staple all plots to the end of this packet.
- Work together!
- My email address jiao@math.utah.edu
- My current office hours: Wed 10 am-12 am WEB 1705
- Additional help for the lab: Lab TA's will be in WEB 1705 M, T, W, Th from 8:30am-1pm. If you go there outside of my office hours, you can ask another section's TA for help. Otherwise, you can ask for help from tutors in the Math Center in the LCB basement.

1. Consider the following operation on functions:

$$S : f(x) \mapsto \frac{f(x) + f(-x)}{2}$$

(a) Compute $S(x^2+1)$, $S(x)$, $S(\sin(x))$, $S(x^2+x-1)$, $S(e^x)$.

(b) Show that the operation S is linear, I.E. the function $S(f+g)$ is the same as the function $S(f)+S(g)$.

(c) Show that solutions of the equation $S(f) = 0$ are exactly the odd functions, $f(-x) = -f(x)$. Which functions solve the equation $S(f) = f$?

(d) Since S is an operation that works on any (!) function, we cannot write it as a linear map on a finite dimensional vector space. Let's restrict our attention to polynomials of degree 3. Compute $S(a_3x^3 + a_2x^2 + a_1x + a_0)$, and use this computation to find a matrix that represents S as a map from the space of degree 3 polynomials to itself. Use the basis $\{1, x, x^2, x^3\}$.

(e) Repeat part d) but use the basis $\{1, 1 + x, 3x^2 + x, x^3 + 2\}$

2. Let M_n be the vector space of all $n \times n$ matrices.
   (a) Is the subset, W of M_n consisting of invertible matrices A, a subspace? If so, prove it. If not, which condition(s) for a subspace does this space violate?

   (b) Is the subset, W of M_n, consisting of symmetric matrices, A, with $A = A^T$ a subspace? If so, prove it. If not, which condition(s) for a subspace does this space violate?

   (c) Find a basis for M_2. What is the dimension of this space?

   (d) Find a basis for the subset of symmetric matrices in M_2 . What is the dimension of this space?

3. Let $\mathcal{B}$ and $\mathcal{C}$ be the following two bases of $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(a) Verify that $\mathcal{B}$ and $\mathcal{C}$ are bases of $\mathbb{R}^3$.

(b) The vector $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ can written as the following linear combination of the basis vectors of $\mathcal{B}$:

$$\mathbf{x} = 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-2) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Show that this is the *only* possible linear combination of the vectors of $\mathcal{B}$ that is equal to $\mathbf{x}$. **Hint:** If you have a different linear combination of the vectors $\mathcal{B}$, what happens when you subtract it from the linear combination shown above?

(c) We define $[\mathbf{x}]_{\mathcal{B}}$ to be "the representation of $\mathbf{x}$ in the basis $\mathcal{B}$", which is the column vector consisting of the coefficients in the linear combination above:

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}.$$

There are matrices B and C so that for any vector $\mathbf{v}$, $\mathbf{v} = B[\mathbf{v}]_{\mathcal{B}}$ and $\mathbf{v} = C[\mathbf{v}]_{\mathcal{C}}$. Find the matrices B and C. (**Hints:** Remember $[\mathbf{x}]_{\mathcal{C}}$ is the vector $\mathbf{x}$ written in terms of the basis vectors of $\mathcal{C}$. Also, what is the relationship between the matrix B and the vectors of $\mathcal{B}$? This should help you in finding C.)

(d) What is $[\mathbf{x}]_{\mathcal{C}}$? (Note that $\mathbf{x} = B[\mathbf{x}]_{\mathcal{B}} = C[\mathbf{x}]_{\mathcal{C}}$. So, you can use the information from part (c) to solve for $[\mathbf{x}]_{\mathcal{C}}$. This procedure is known as a change of basis, because you are given a vector described in terms of some set of basis vectors, and you want to write the same vector described in terms of some new set of basis vectors.)