Math 2250: Chapter 4 Glossary of Terms for Vector Spaces

Vector space

A vector space V is a collection of vectors such that:
(i) V is closed under addition; that is, given the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V, the vector $\mathbf{v}_1 + \mathbf{v}_2$ is also in V.
(ii) V is closed under scalar multiplication; that is, given a number c in $\mathbb{R}$ and a vector $\mathbf{v}$ in V, we have the vector $c\mathbf{v}$ is in V.[1]
Ex1. $\mathbb{R}^n$ and $\mathbb{C}^n$ are vector spaces.
Ex2. Solutions to homogeneous linear differential equations form a vector space.
Vector spaces are infinite sets, excepting a few trivial examples.

Subspace

W is a **subspace** of a given a vector space V if W is contained in V and W is itself a vector space.
To test if a set W a subspace of V, it must be that
(i) The sum of any two vectors in W is also in W
(ii) Any scalar multiple of a vector in W is also in W.
Ex1. Any plane in $\mathbb{R}^3$ passing through $\vec{0}$ is a subspace.
Ex2. The trivial subspace contains only the $\mathbf{0}$ vector.
Ex4. Any vector space is a subspace of itself.
Ex4. A subspace of $\mathbb{R}^2$ is the set of all vectors of the form $[0 \;\; a]^T$ where a is any real number. This subspace is all vectors where the x-component is 0.

Linear combination

Given a set of vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ and a set of scalars $c_1, c_2, ..., c_k$ a **linear combination** of these vectors and scalars is:

$$c_1\mathbf{v} + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k.$$

Linear combinations allow the creation of new vectors from a given set of vectors using the vector algebra of summation and scalar multiplication.
Ex. We can express the vector $[2 \;\; 3]^T$ as a linear combination of the standard basis of $\mathbb{R}^2$ with $c_1 = 2$ and $c_2 = 3$. That is:

$$2\begin{bmatrix}0\\1\end{bmatrix} + 3\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}.$$

Linear independence and dependence

Linear independence and dependence is a property that describes sets of vectors $\{\mathbf{v}_1, \mathbf{v}_2, .., \mathbf{v}_k\}$.
A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, .., \mathbf{v}_k\}$ is **linearly independent** if the only solution to the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k = \mathbf{0}$$

is trivial, that is, $c_1 = c_2 = ... = c_k = 0$.
The set of vectors is **linearly dependent** if it is *not* linearly independent.
Ex. The following set of vectors

$$\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$$

is linearly independent because the only c_1 and c_2 values that satisfy the equation

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

are $c_1 = 0$ and $c_2 = 0$.
Conversely, the vectors

$$\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-2\\-2\end{bmatrix}\right\}$$

[1] c could be a complex number

are linearly dependent because the equation

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

can be solved by setting $c_1 = 2$ and $c_2 = 1$, which are non-zero. Note the above can also be solved by setting $c_1 = 0$ and $c_2 = 0$, but it is the existence of other non-zero solutions that define the set of vectors as dependent.

Span

Let $\{\mathbf{v}_1, \mathbf{v}_2, .., \mathbf{v}_k\}$ be a set vectors in a vector space V, then the set W of all linear combinations of $\{\mathbf{v}_1, \mathbf{v}_2, .., \mathbf{v}_k\}$ is their span.
Ex. The span of the set of vectors

$$S = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

is $\mathbb{R}^2$. That is, we write $span(S) = \mathbb{R}^2$.
Note that the span is an infinite set generated from a set of discrete vectors $\{\mathbf{v}_1, \mathbf{v}_2, .., \mathbf{v}_k\}$. Note that "span" is used as a noun and also sometimes as a verb. One might say the "span of the vectors is ..." (noun) as above, or one could say "the vectors span $\mathbb{R}^2$" (verb).

Basis

A **basis** for a vector space V, is a set of S vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ that are (i) linearly independent and (ii) span V; that is $span(S) = V$.
Ex. The following vectors form a basis for $\mathbb{R}^2$:

$$S = \left\{ \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \end{bmatrix} \right\}$$

Note that a basis is a special type of a discrete set of vectors S that generates a given vector space—an infinite set. The set S generates (i.e., spans) V with a minimal number of vectors due to the fact that S is linearly independent.

Dimension

The number n of vectors in a basis for a vector space V is the dimension of V, written $dim(V) = n$.
ex. $\mathbb{R}^2$ is 2-dimensional because every basis of $\mathbb{R}^2$ contains only $n = 2$ vectors.

Standard basis of $\mathbb{R}^n$

The **standard basis of** $\mathbb{R}^n$ is the set of unit vectors pointing in the direction of axes on a Cartesian coordinate system. That is, the set of vectors $\mathbf{v}_i$ where there is a 1 in the i^{th} entry and 0's everywhere else.
Ex. The standard basis for $\mathbb{R}^2$ is:

$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$