

This test is:

- closed-book
- closed-notes
- no-calculator
- 80 minutes

Indicate your answers clearly, and show your work. Partial credit will be awarded based on work shown. Full credit will not be awarded without some work shown.

Fun fact of life: if your work is not legible, I will not be able to read it. The ramifications of this outcome should be clear.

There are 3 questions with multiple parts; each question is worth a total of 20 points.

All pages are one-sided. If on any problem you require more space, use the back of the page.

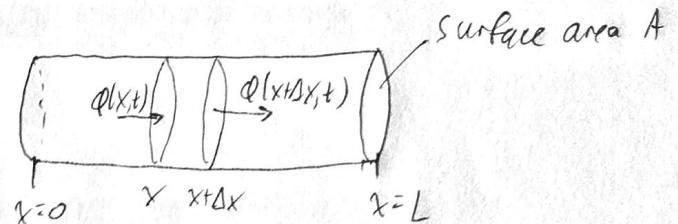
DO NOT TURN THIS PAGE UNTIL DIRECTED TO BEGIN

1. (20 pts) Derive the heat equation for a one-dimensional rod of length L assuming constant thermal properties and no sources. (You may ignore any initial and boundary conditions.)

$u(x, t)$: temperature

$\phi(x, t)$: heat flux per unit area

$e(x, t)$: energy density



Conservation of energy: $\left(\frac{\text{rate of change}}{\text{in energy}} \right) = \left(\begin{array}{l} \text{heat energy} \\ \text{flow across} \\ \text{boundaries} \end{array} \right) + \left(\begin{array}{l} \text{heat generated} \\ \text{or lost} \\ \text{internally} \end{array} \right)$

"no sources" \Rightarrow this is 0.

Applied to rod between x and $x+\Delta x$:

$$\frac{\partial}{\partial t} [e(x, t) A \Delta x] = +A \phi(x, t) - A \phi(x+\Delta x, t)$$

$$\frac{\partial e}{\partial t} = \frac{\phi(x, t) - \phi(x+\Delta x, t)}{\Delta x}$$

$$\text{as } \Delta x \rightarrow 0, \text{ right-hand side} \approx -\frac{\partial \phi}{\partial x}$$

so as $\Delta x \rightarrow 0$, conservation of energy reads

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}$$

Temperature u and energy e related by specific heat and mass density.

"Constant thermal properties" \Rightarrow specific heat c and density ρ are constants.

$$e(x, t) = c \rho u(x, t)$$

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Heat flux ϕ and temperature u related through Fourier's law of heat conduction:

$$\phi(x,t) = -K_o \cdot \frac{\partial u}{\partial x}, \quad K_o: \text{thermal conductivity} \\ (\text{again, constant}).$$

We have the equations: $\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}$

$$e(x,t) = c_p u(x,t)$$

$$\phi(x,t) = -K_o \frac{\partial u}{\partial x}$$

Using the last two in' the first equation:

$$\frac{\partial}{\partial t} [c_p u(x,t)] = -\frac{\partial}{\partial x} [-K_o \frac{\partial u}{\partial x}]$$

$$\frac{\partial u}{\partial t}(x,t) = \frac{K_o}{c_p} \frac{\partial^2 u}{\partial x^2}(x,t)$$

This is the heat equation.

2. (20 pts) Solve the following eigenvalue problem: find all eigenvalues λ and eigenfunctions $\phi(x)$. You must show all work, including exhausting all possible values of λ .

$$\begin{aligned}\phi''(x) + \lambda\phi(x) &= 0, & 0 < x < L \\ \phi(0) &= 0, & \phi'(L) = 0\end{aligned}$$

Solution properties change depending on sign of λ , so break up search to cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$.

$\lambda < 0$: $\phi''(x) - |\lambda|\phi(x) = 0$

$$\phi(0) = \phi'(L) = 0$$

roots of characteristic equation: $r_1 = \sqrt{|\lambda|}$, $r_2 = -r_1 = -\sqrt{|\lambda|}$

$$\phi(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

$$\phi(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$$

$$\phi'(L) = 0 \Rightarrow c_1 r_1 e^{r_1 L} + c_2 r_2 e^{r_2 L} = 0$$

$$c_1 r_1 e^{r_1 L} - c_2 r_1 e^{-r_1 L} = 0$$

$$c_1 r_1 (e^{r_1 L} + e^{-r_1 L}) = 0$$

$$r_1 = \sqrt{|\lambda|} > 0$$

$$e^{r_1 L} > 0 \text{ and } e^{-r_1 L} > 0$$

$$\text{So } c_1 = 0 \Rightarrow c_2 = 0$$

$\phi(x) = 0$ is the only solution, so there are no negative eigenvalues.

$\lambda = 0$: $\phi''(x) = 0 \rightarrow \phi(x) = c_1 + c_2 x$

$$\phi(0) = \phi'(L) = 0$$

$$\begin{aligned}\phi(0) = 0 &\Rightarrow c_1 = 0 \\ \phi'(L) = 0 &\Rightarrow c_2 = 0\end{aligned} \rightarrow \phi(x) = 0 \text{ so } \lambda = 0 \text{ not an eigenvalue}$$

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$$\lambda > 0 \quad \varphi''(x) + \lambda \varphi(x) = 0$$

$$\varphi(0) = \varphi'(L) = 0$$

roots of characteristic equation: $r_1 = i\sqrt{\lambda}$, $r_2 = -i\sqrt{\lambda}$

$$\varphi(x) = c_1 \cos(x\sqrt{\lambda}) + c_2 \sin(x\sqrt{\lambda})$$

$$\varphi(0) = 0 \Rightarrow c_1 = 0$$

$$\varphi'(L) = 0 \Rightarrow c_2 \sqrt{\lambda} \cos(L\sqrt{\lambda}) = 0$$

$$\sqrt{\lambda} > 0, \text{ so } \cos(L\sqrt{\lambda}) = 0$$

$$L\sqrt{\lambda} = \left(\frac{2n-1}{2}\right)\pi, \quad n=1, 2, \dots$$

$$\lambda_n = \left[\left(\frac{2n-1}{2}\right)\frac{\pi}{L}\right]^2, \quad n=1, 2, \dots$$

$$\underline{\varphi_n(x) = \sin(x\sqrt{\lambda_n}) = \sin\left(\frac{(2n-1)\pi}{2L}x\right), \quad n=1, 2, \dots}$$

3. (20 pts) Compute the solution $u(x, t)$ to the following one-dimensional heat equation:

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0$$

subject to the initial and boundary conditions

$$\begin{aligned} u(x, 0) &= 1 - (1 - x)^2, \\ u(0, t) &= 0, \\ \frac{\partial u}{\partial x}(1, t) &= 0 \end{aligned}$$

Show all work. Your solution must be written down in terms of explicit, computable expressions or integrals, but you need not compute the values of these integrals. You may use any results derived from previous problem(s).

Separation of variables: $u(x, t) = \phi(x) T(t)$

$$u_t = u_{xx} \implies \phi(x) T'(t) = \phi''(x) T(t)$$

~~$$\text{then } \frac{\phi''(x)}{\phi(x)} = \frac{T'(t)}{T(t)} = -\lambda \quad (\text{unknown constant})$$~~

$$u(0, t) = 0 \implies \phi(0) \cdot T(t) = 0$$

nontrivial solution $\Rightarrow \phi(0) = 0$

$$\text{Similarly, } \frac{\partial u}{\partial x}(1, t) = 0 \implies \phi'(1) = 0$$

Equation for $\phi(x)$:

$$\begin{cases} \phi''(x) + \lambda \phi(x) = 0 \\ \phi(0) = 0, \quad \phi'(1) = 0 \end{cases}$$

Nontrivial solutions dictated by solution to eigenvalue problem in question #2:

$$\lambda_n = \left[\frac{(2n-1)\pi}{L} \right]^2, \quad \phi_n(x) = \sin \left(\left(\frac{2n-1}{2} \right) \pi x \right), \quad n=1, 2, \dots$$

Orthogonality: $\int_0^L \phi_n(x) \phi_m(x) dx = \begin{cases} 0, & n \neq m \\ L/2, & n = m \end{cases}$

Since $L=1$ in this problem: $\lambda_n = \left[\frac{(2n-1)\pi}{2} \right]^2, \quad \phi_n = \sin \left(\left(\frac{2n-1}{2} \right) \pi x \right)$

$$\int_0^1 \phi_n(x) \phi_m(x) dx = \begin{cases} 0, & n \neq m \\ 1/2, & n = m \end{cases}$$

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Equation for $T(t)$ with $\lambda = \lambda_n$:

$$T'_n(t) + \lambda_n T_n(t) = 0$$

$$T_n(t) = c_n \exp(-\lambda_n t) = c_n \exp\left(-\left[\frac{(2n-1)\pi}{2}\right]^2 t\right)$$

$$\text{For } \lambda = \lambda_n: u_n(x, t) = \phi_n(x) T_n(t) = c_n \exp\left(-\left[\frac{(2n-1)\pi}{2}\right]^2 t\right) \sin\left(\left[\frac{(2n-1)\pi}{2}\right] x\right)$$

$$\text{Superposition: } u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} c_n \exp(-\lambda_n t) \sin(\sqrt{\lambda_n} x)$$

$$\text{Initial condition: } u(x, 0) = \sum_{n=1}^{\infty} c_n \exp(0) \cdot \sin(\lambda_n x)$$

$$= \sum_{n=1}^{\infty} c_n \phi_n(x)$$

$$1 - (1-x)^2 = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

$$\text{Orthogonality} \Rightarrow \int_0^1 [1 - (1-x)^2] \phi_m(x) dx = c_m \cdot \frac{1}{2}$$

$$\left\{ \begin{array}{l} c_m = 2 \int_0^1 [1 - (1-x)^2] \sin\left(\frac{(2m-1)\pi}{2} x\right) dx \\ u(x, t) = \sum_{n=1}^{\infty} c_n \phi_n(x) T_n(t) \\ \phi_n(x) = \sin(\sqrt{\lambda_n} x) \\ T_n(t) = \exp(-\lambda_n t) \\ \lambda_n = \left(\frac{2n-1}{2}\pi\right)^2 \end{array} \right.$$

Solution