Department of Mathematics, University of Utah
Analysis of Numerical Methods I
MTH6610 – Section 001 – Fall 2017

Lecture notes – Numerical differentiation 2
Friday, December 1, 2017

These notes are not a substitute for class attendance. Their main purpose is to provide a lecture overview summarizing the topics covered.

Recall from last time that we sought to develop approximate formulas for computing derivatives of functions:

$$f^{(q)}(x) \approx \sum_{j=1}^{N} w_j f(x_j), \qquad x, x_1, \ldots, x_N \in [a,b], \tag{1}$$

for $q = 0, \ldots, N-1$. By using Taylor's Theorem evaluated at x_j expanding around the point x, we determined that the weights w_j are given by solutions to the linear system

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ (x_1 - x) & (x_2 - x) & (x_3 - x) & \cdots & (x_N - x) \\ (x_1 - x)^2 & (x_2 - x)^2 & (x_3 - x)^2 & \cdots & (x_N - x)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (x_1 - x)^{N-1} & (x_2 - x)^{N-1} & (x_3 - x)^{N-1} & \cdots & (x_N - x)^{N-1} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_N \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ q! \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2}$$

where the non-zero entry on the right-hand side is in the $(q+1)$st entry. In this way we obtain a finite difference rule that is accurate to order $N - q$.
However, we could use an alternative approach: instead we could form a polynomial interpolant of f on the points $x_1, \ldots, x_N$, differentiate it, and finally evaluate it at x. In other words, if $\ell_j(x)$ are the cardinal Lagrange interpolating polynomials associated to the points $x_1, \ldots, x_N$, then we could form the approximation

$$\begin{aligned} f^{(q)}(x) &\approx p_{N-1}^{(q)}(x) = \frac{\mathrm{d}^q}{\mathrm{d}x^q} \sum_{j=1}^{N} f(x_j)\ell_j(x) = \sum_{j=1}^{n} f(x_j)\ell_j^{(q)}(x) = \sum_{j=1}^{n} v_n f(x_j), \\ v_j &= \ell_j^{(q)}(x) \end{aligned}$$

Since the polynomial interpolant p_{N-1} is accurate to order $(b-a)^N$, we expect that, after taking q derivatives, we obtain a finite difference formula with the weights v_j that is accurate to order $N - q$, the same as for the w_j. We will show here that these two approaches generate exactly the same rule.
Recall our notation: x and x_j are all fixed points on $[a,b]$. In the following we will use y as the independent variable of approximation. We first compute an expansion of the cardinal

Lagrange interpolants in the basis $(y-x)^k$:

$$P_{N-1} = \text{span}\left\{1, y-x, (y-x)^2, \ldots, (y-x)^{N-1}\right\} \quad \Longrightarrow \quad \ell_j(x) = \sum_{k=1}^{N} d_{j,k}(y-x)^{k-1}.$$

The existence and uniqueness of the coefficients $d_{j,k}$ is guaranteed by unisolvence of polynomial interpolation since $\ell_j \in P_{N-1}$. We can explicitly construct these coefficients by solving the interpolation problem using the basis $(y-x)^{k-1}$:

$$\boldsymbol{V}\boldsymbol{d}_j = \begin{pmatrix} 1 & (x_1 - x) & (x_1-x)^2 & \cdots & (x_1-x)^{N-1} \\ 1 & (x_2 - x) & (x_2-x)^2 & \cdots & (x_2-x)^{N-1} \\ 1 & (x_3 - x) & (x_3-x)^2 & \cdots & (x_3-x)^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (x_N - x) & (x_N-x)^2 & \cdots & (x_N-x)^{N-1} \end{pmatrix} \begin{pmatrix} d_{j,1} \\ d_{j,2} \\ d_{j,3} \\ \vdots \\ d_{j,N} \end{pmatrix} = \boldsymbol{e}_j,$$

where $\boldsymbol{e}_j$ the cardinal unit vector in the jth direction. (Recall that $\ell_j(x_k) = \delta_{j,k}$.) The collection of systems above for $j = 1, \ldots, N$, implies that the vector $\boldsymbol{d}_j$ is simply the jth column of $\boldsymbol{V}^{-1}$:

$$(\boldsymbol{d}_1 \ \ \boldsymbol{d}_2 \ \ \cdots \ \ \boldsymbol{d}_N) = \boldsymbol{V}^{-1}.$$

The weights v_j are computed as

$$v_j = \ell_j^{(q)}(x) = \left.\left(\frac{d^q}{dy^q}\sum_{k=1}^{N} d_{j,k}(y-k)^{k-1}\right)\right|_{y=x} = d_{j,q+1}q!$$

This shows that

$$v_j = \boldsymbol{e}_{q+1}^T \boldsymbol{d}_j q! \quad \Longrightarrow \quad \boldsymbol{v}^T = (v_1, \ldots, v_N) = \boldsymbol{e}_{q+1}^T \boldsymbol{V}^{-1} q!,$$

Or:

$$\boldsymbol{v} = \boldsymbol{V}^{-T}\boldsymbol{e}_{q+1}q!.$$

Note that the matrix in (2) is $\boldsymbol{V}^T$ and the right-hand side is $\boldsymbol{e}_{q+1}q!$, so that we also have

$$\boldsymbol{w} = \boldsymbol{V}^{-T}\boldsymbol{e}_{q+1}q!.$$

Thus, as claimed, the weights produced via a Taylor series approach are the same as those produced via a polynomial interpolation approach.