Department of Mathematics, University of Utah
Analysis of Numerical Methods I
MTH6610 – Section 001 – Fall 2017

Lecture notes – Quadrature
Monday November 27, 2017

These notes are not a substitute for class attendance. Their main purpose is to provide a lecture overview summarizing the topics covered.

Reading: Isaacson & Keller, Section 7.1

Quadrature is the numerical approximation of definite integrals. The prototypical example is a one-dimensional integral over a compact interval:

$$\int_a^b f(x)\,\mathrm{d}x, \qquad -\infty < a < b < \infty.$$

Perhaps the most common type of quadrature uses N point evaluations of f to approximate this integral:

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{j=1}^N w_j f(x_j),$$

where the x_j are the quadrature *abscissae* or *nodes* and the w_j are the quadrature *weights*, both of which must be prescribed or determined.
A popular approach to constructing such quadrature rules is to use polynomial interpolation to construct *interpolatory* quadrature rules. I.e., if one prescribes $x_1, \ldots, x_N$ as distinct points in $[a, b]$, then the unique degree-$(N-1)$ polynomial that interpolates $f(x)$ at these points is given by

$$p_{N-1}(x) = \sum_{j=1}^N f(x_j)\ell_j(x),$$

where ℓ_j are the N cardinal Lagrange interpolating polynomials associated with the points $x_1, \ldots, x_N$. An interpolatory quadrature rule constructs a quadrature rule via the approximation

$$\int_a^b f(x)\,\mathrm{d}x \approx \int_a^b p_{N-1}(x)\,\mathrm{d}x = \sum_{j=1}^N f(x_j) \int_a^b \ell_j(x)\,\mathrm{d}x,$$

which shows that we can define the quadrature weights as

$$w_j = \int_a^b \ell_j(x)\,\mathrm{d}x.$$

This provides a formulaic procedure for computing interpolatory quadrature rules once $x_1, \ldots, x_N$ are prescribed.

There is a different, equivalent procedure for computing the weights w_j of an interpolatory quadrature rule for the given nodes $x_1, \ldots, x_N$. We note that

$$f \in \text{span}\left\{1, x, x^2, \ldots, x^{N-1}\right\} \quad \Rightarrow \quad f(x) = p_{N-1}(x),$$

so that an interpolatory quadrature rule with the weights w_j satisfies

$$\int_a^b p(x)\,\mathrm{d}x = \sum_{j=1}^N p(x_j) w_j, \qquad\qquad p \in \text{span}\left\{1, x, x^2, \ldots, x^{N-1}\right\}.$$

Noting that the above constraint is linear in the unknowns w_j, then let $q_1, \ldots, q_N$ be any basis for $\text{span}\left\{1, x, \ldots, x^{N-1}\right\}$, so that the vector solution $\boldsymbol{w} \in \mathbb{R}^N$ to the linear system

$$\boldsymbol{V}\boldsymbol{w} = \boldsymbol{b}, \qquad\qquad (V)_{k,j} = q_k(x_j), \qquad\qquad (b)_k = \int_a^b q_k(x)\,\mathrm{d}x,$$

contains the interpolatory quadrature rule weights w_j. Solving the above linear system is an equivalent way of computing weights w_j for an interpolatory rule.

The error committed by an interpolatory quadrature rule can be understood by considering the error committed by the interpolation. The following is a standard error formula committed by polynomial interpolation on an interval $[a, b]$:

$$f(x) - p_{N-1}(x) = \prod_{j=1}^N (x - x_j) \frac{f^{(N)}(\xi)}{N!}, \qquad\qquad \xi = \xi(x) \in [a, b].$$

The number ξ is usually not computable. However, noting that

$$\left| \prod_{j=1}^N (x - x_j) \right| \le (b-a)^N,$$

then

$$\max_{x \in [a,b]} \left| f(x) - p_{N-1}(x) \right| \le \frac{(b-a)^N}{N!} \max_{x \in [a,b]} \left| f^{(N)}(x) \right|,$$

so that the error committed by the interpolatory quadrature rule is

$$\left| \int_a^b f(x)\,\mathrm{d}x - \int_a^b p_{N-1}(x)\,\mathrm{d}x \right| \le (b-a) \max_{x\in[a,b]} \left| f(x) - p_{N-1}(x) \right| \le \frac{(b-a)^{N+1}}{N!} \max_{x \in [a,b]} \left| f^{(N)}(x) \right|.$$

One frequently uses interpolatory quadrature rules in a *composite* (i.e,. piecewise) form, so that the interval length $(b-a)$ is usually small, $(b-a) < 1$. Under this assumption, one then sees that this provides a convergent interpolatory estimate, so long as the Nth derivative of f does not become too large. In particular, the error scales as $(b-a)^{N+1}$ as $(b-a) \to 0$. We say that the *order* of the quadrature rule is $N+1$.

A special family of interpolatory quadrature rules is given by the Newton-Cotes rules. These are N-point ($N \ge 2$) rules using equidistant points on $[a, b]$. If the nodes include the endpoints, these are called *closed* Newton-Cotes rules. If they do not include the endpoints, they are open Newton-Cotes rules.

Finally, we recall that interpolation on equidistant nodes is usually a bad idea for high-degree polynomial interpolation. Therefore, the Newton-Cotes rules are only viable when $N \lesssim 10$.