Department of Mathematics, University of Utah

Analysis of Numerical Methods I

MTH6610 – Section 001 – Fall 2017

Lecture notes – Fourier Series

Friday November 10, 2017

These notes are not a substitute for class attendance. Their main purpose is to provide a lecture overview summarizing the topics covered.

We consider the approximation of functions on finite intervals via a Fourier Series. This requires some notation. We consider the interval $[0, 2\pi]$, and introduce the space of square-integrable perioidic functions:

$$L^2 = L^2_p\left([0, 2\pi]; \mathbb{C}\right) = \left\{ f : [0, 2\pi] \to \mathbb{C} \;\middle|\; \|f\|_{L^2} < \infty, \text{ and } f(0) = f(2\pi) \right\},$$

where

$$\|f\|^2_{L^2} = \langle f, f \rangle, \qquad\qquad \langle f, g \rangle = \int_0^{2\pi} f(x) g(x) \,\mathrm{d}x$$

We will also need integer Sobolev spaces; for $s = 0, 1, \ldots,$, define

$$H^s = H^s_p\left([0, 2\pi]; \mathbb{C}\right) = \left\{ f \in L^2 \;\middle|\; f^{(r)} \in L^2, r = 0, \ldots, s, \text{ and } f^{(r)}(0) = f^{(r)}(2\pi), r = 0, \ldots, s-1 \right\},$$

where $f^{(s)}$ is the sth derivative of f. Note that $L^2 = H^0$. We define the norm on this space as

$$\|f\|^2_{H^s} = \sum_{k=0}^{s} \left\| f^{(k)} \right\|^2_{L^2}.$$

Define the L^2-orthonormal functions

$$v_j(x) = \frac{1}{\sqrt{2\pi}} e^{ijx}, \qquad\qquad j \in \mathbb{Z},$$

where $i = \sqrt{-1}$ is the imaginary unit. The family $\{v_j\}_{j \in \mathbb{Z}}$ is complete in L^2. Therefore,

$$f \in L^2 \quad \Longrightarrow \quad f(x) = \sum_{j \in \mathbb{Z}} \widehat{f}_j v_j(x),$$

where $\widehat{f}_j$ are some constants. Parseval's identity states

$$\|f\|^2_{L^2} = \sum_{j \in \mathbb{Z}} \left| \widehat{f}_j \right|^2.$$

Formally taking inner products on both sides of the expansion of $f(x)$ shows that

$$\widehat{f}_j = \langle f, v_j \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{ijx} \,\mathrm{d}x$$

When using f in computational approximations, we frequently represent it as a truncated version of its infinite series. This has the form

$$f_N(x) = \sum_{|j|<N} \widehat{f}_j v_j(x), \qquad\qquad N \in \mathbb{N}.$$

A natural question is how large of an error we make in approximating f by f_N. To answer this question, first suppose that $f \in H^s$ for some $s > 0$. We have computed the expansion coefficients $\widehat{f}_j$ of f, but we can likewise compute the expansion coefficients $\widehat{f}_j^{(1)}$ of $f^{(1)}$:

$$f \in H^1 \quad \Longrightarrow \quad f^{(1)} \in L^2 \Longrightarrow \quad f^{(1)} = \sum_{j\in\mathbb{Z}} \widehat{f}_j^{(1)} e^{ijx},$$

where

$$\widehat{f}_j^{(1)} = \left\langle f^{(1)}, v_j \right\rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f^{(1)} e^{ijx} \, \mathrm{d}x.$$

A usage of integration by parts yields the identity

$$\widehat{f}_j^{(1)} = ij \widehat{f}_j \qquad \Longrightarrow \qquad \left|\widehat{f}_j\right|^2 = \frac{1}{j^2} \left|\widehat{f}_j^{(1)}\right|^2.$$

This identity will be useful in bounding the error committed by replacing f by f_N, since

$$\begin{aligned}
\|f - f_N\|_{L^2}^2 &= \sum_{|j|\geq N} \left|\widehat{f}_j\right|^2 = \sum_{|j|\geq N} \frac{1}{j^2} \left|\widehat{f}_j^{(1)}\right|^2 \leq \frac{1}{N^2} \sum_{|j|\geq N} \left|\widehat{f}_j^{(1)}\right|^2 \\
&\leq \frac{1}{N^2} \sum_{j\in\mathbb{Z}} \left|\widehat{f}^{(1)}\right|^2 \leq \frac{1}{N^2} \left\|f^{(1)}\right\|_{L^2}^2 \leq \frac{1}{N^2} \|f\|_{H^1}^2 .
\end{aligned}$$

Repeating this argument by induction yields the following classical approximation inequality:

$$f \in H^s \qquad \Longrightarrow \qquad \|f - f_N\|_{L^2} \leq N^{-s} \|f\|_{H^s} .$$

This statement has a simple interpretation: *smoothness* of f, i.e., existence of sufficiently many derivatives, implies faster convergence when approximating by Fourier Series.