Department of Mathematics, University of Utah

Analysis of Numerical Methods I

MTH6610 – Section 001 – Fall 2017

Lecture notes – Eigenvalues

Wednesday October 25, 2017

These notes are not a substitute for class attendance. Their main purpose is to provide a lecture overview summarizing the topics covered.

Reading: Trefethen & Bau III, Lectures 24

Let $A \in \mathbb{C}^{n\times n}$. A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of A if there exists a nonzero vector v such that

$$Av = \lambda v$$

For a fixed eigenvalue λ_j, the subspace $V_j \subset \mathbb{C}^n$ containing vectors v satisfying the above equation is called the eigenspace associated to λ_j. The eigenspace V_j has dimension d_j, and this dimension is called the *geometric multiplicity* of the eigenvalue λ_j.

Let $\lambda_1, \ldots, \lambda_p$ be an enumeration of the eigenvalues of A, and let λ_j have corresponding eigenspace V_j of dimension $d_j \geq 1$. Let $v_{j1}, \ldots, v_{jd_j}$ be any basis for V_j. Then we have the matrix equality

$$AV = V\Lambda,$$

where the matrices V and Λ are defined as

$$\Lambda = \begin{pmatrix} \lambda_1 I_{d_1} & & & \\ & \lambda_2 I_{d_2} & & \\ & & \ddots & \\ & & & \lambda_p I_{d_p} \end{pmatrix}, \quad V = \begin{pmatrix} v_{11} & \cdots & v_{1d_1} & v_{21} & \cdots & v_{2d_2} & \cdots & v_{pd_p} \end{pmatrix}$$

If λ is an eigenvalue of A, the definition implies that

$$p_A(\lambda) := \det(\lambda I - A) = 0,$$

and also any λ satisfying the above is an eigenvalue of A. The Laplace expansion of the determinant implies that $z \mapsto p_A(z)$ is a polynomial of degree n. p_A is the called the *characteristic polynomial* of A. We see that A must therefore have exactly n eigenvalues corresponding to the n roots of p_A, some of which may be repeated. The multiplicity of a root λ of p_A is called the *algebraic multiplicity* of the eigenvalue λ.

Given a square matrix A and an invertible matrix S, a similarity transformation applied to A is the map $A \mapsto S^{-1}AS$. Similarity transformations preserve eigenvalues, algebraic multiplicities, and geometric multiplicities.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, with repeated values for algebraic multiplicity greater than 1. By using properties of the chacateristic polynomial, we have that

$$\det A = \prod_{j=1}^{n} \lambda_j, \qquad \mathrm{trace}(A) = \sum_{j=1}^{n} \lambda_j.$$

The geometric multiplicity of an eigenvalue is at most the algebraic multiplicity of that eigenvalue. Any eigenvalue whose geometric multiplicity is strictly less than its algebraic multiplicity is called a *defective eigenvalue*. Any matrix with a defective eigenvalue is called a defective matrix.

When a matrix is defective, one cannot form an invertible matrix V of its eigenvalues. When a matrix is *non*defective, then V is invertible and we can form the eigenvalue decomposition

$$A = V \Lambda V^{-1},$$

and in such cases we say that A is *diagonalizable*, meaning that it is similar to a diagonal matrix. While not all matrices have eigenvalue decompositions (i.e., defective ones do not), all matrices do have a Jordan decomposition $A = VJV^{-1}$, where J has entries only the main and super-diagonal.

A special class of matrices are those who are diagonalizable via a unitary similar transform, $V^{-1} = V^*$. We have already seen that Hermitian matrices fall into this class, but the more general class of matrices are *normal* matrices. A matrix A is a normal matrix if it commutes with its conjugate transpose, $AA^* = A^*A$. A matrix is a normal matrix if and only if it is diagonalizable via a unitary matrix.

While not every matrix is unitarilty diagonalizable (i.e., non-normal matrices are not), all matrices can be brought into upper triangular form via a unitary transformation:

$$A = UTU^*,$$

where U is unitary and T is upper triangular. This is called the *Schur decomposition*. This decomposition plays a fundamental role in numerical algorithms: since A is similar to T, they share the same eigenvalues. Since T is triangular, its eigenvalues can be read off from the diagonal. This provides one of the more well-conditioned strategies for computing eigenvalues: compute eigenvalues of A from its Schur factor T, which can be computed via unitary transformations of A.