Department of Mathematics, University of Utah
Analysis of Numerical Methods I
MTH6610 – Section 001 – Fall 2017

Lecture notes: Conditioning
Friday, September 22, 2017

These notes are not a substitute for class attendance. Their main purpose is to provide a lecture overview summarizing the topics covered.

Reading: Trefethen & Bau III, Lectures 12
Reading: Isaacson & Keller, Sections 1.3

We have seen that some algorithms are sensitive to small changes in inputs to the problem. However, this sensitivity could stem from one of two things: (1) either the inherent sensitivity of the exact mathematical problem to inputs, or (2) ill-behavior of algorithms that operate in finite-precision arithmetic. Here we consider codify the first of these issues, which is called the *conditioning* of a problem.
Consider a general (not necessarily linear) function $f : \mathbb{C}^n \to \mathbb{C}^m$, with some given norms on $\mathbb{C}^n$ and $\mathbb{C}^m$. We will denote both of these norms as $\|\cdot\|$. At some given $x \in \mathbb{C}^n$, we are interested in studying the sensitivity of $f(x)$ with respect to small perturbations of the input x. A reasonable measure of this sensitivity is the relative error when we perturb x by δx,

$$\frac{\|f(x+\delta x) - f(x)\|}{\|\delta x\|}.$$

Making the notational definition $\delta f = f(x+\delta x)$, then the absolute condition number of f at x is the worst relative error when infinitesimal perturbations are made:

$$\widehat{\kappa}(x) = \lim_{\delta \to 0} \sup_{\|\delta x\| \leq \delta} \frac{\|\delta f\|}{\|\delta x\|}.$$

Note that this definition is a property of the function f itself, and *not* of any algorithm that implements f. Since x and $f(x)$ may actually be quite large in norm themselves, it is frequently more appropriate to work with the relative condition number, which normalizes by these values,

$$\kappa(x) = \lim_{\delta \to 0} \sup_{\|\delta x\| \leq \delta} \frac{\|\delta f\| \|x\|}{\|f(x)\| \|\delta x\|}$$

If f is a smooth function of its inputs, then

$$\widehat{\kappa}(x) = \|J(x)\|, \qquad\qquad J(x) = \frac{\partial f}{\partial x}(x)$$

where $J \in \mathbb{C}^{m \times n}$ is the Jacobian matrix of f. Here, $\|J\|$ is the matrix norm induced by the given norms on $\mathbb{C}^m$ and $\mathbb{C}^n$.
These definitions immediately allow us to consider the conditioning of some familiar linear problems.

- Let $A \in C^{n\times n}$ be nonsingular and $x \in \mathbb{C}^n$ be given. Evaluating $y = Ax$ for perturbations in x leads us to consider the function $x \mapsto Ax$. This function has relative condition number $\kappa(x) \le \|A\|\|A^{-1}\|$. Equality is achieved for certain x vectors that depend on A. This result also applies to the problem $b \mapsto A^{-1}b$ for given b.

- Let $A \in C^{n\times n}$ be nonsingular and $b \in \mathbb{C}^n$ be given. Evaluating $y = Ax$ for perturbations in A leads us to conside the function $A \mapsto A^{-1}b$. This function has relative condition number $\kappa(x) = \|A\|\|A^{-1}\|$.

For some fixed nonsingular matrix A, the quantity $\kappa(A) = \|A\|\|A^{-1}\|$ is called the condition number of the matrix A. When A is rectangular and we use the induced 2-norm $\|\cdot\|$, we can use $\kappa(A) = \|A\|\|A^{+}\| = \sigma_1(A)/\sigma_r(A)$, where $r = \text{rank}(A)$.