Department of Mathematics, University of Utah
Analysis of Numerical Methods I
MTH6610 – Section 001 – Fall 2017

Lecture notes
Friday September 8, 2017

These notes are not a substitute for class attendance. Their main purpose is to provide a lecture overview summarizing the topics covered.

Reading: Trefethen & Bau III, Lecture 6

We consider projection matrices. A matrix $P \in \mathbb{C}^{n \times n}$ is a projection matrix if and only if it is *idempotent*:

$$P^2 = P$$

A projection matrix is an *orthogonal projector* if, in addition to being idempotent,

$$P = P^*$$

A projection matrix that is not orthogonal is called *oblique*. If P is a(n orthogonal) projection matrix, then $I - P$ is also a(n orthogonal) projection matrix.
Projection matrices are uniqely determined by their range and their kernel. In particular,

$$P^2 = P \qquad \Longrightarrow \qquad \mathrm{range}(P) \oplus \ker(P) = \mathbb{C}^n.$$

In fact, if $\mathcal{V} \subset \mathbb{C}^n$ and $\mathcal{W} \subset \mathbb{C}^n$ satisfy $\mathcal{V} + \mathcal{W} = \mathbb{C}^n$ and $\dim \mathcal{V} + \dim \mathcal{W} = n$ (hence $\mathcal{V} \oplus \mathcal{W} = \mathbb{C}^n$), then there is a unique projection matrix P with range $\mathcal{V}$ and kernel $\mathcal{W}$. When P is an orthogonal projector, then $\mathrm{range}(P) \perp \ker(P)$.
Given $\mathcal{V}$ and $\mathcal{W}$ satisfying $\mathcal{V} \oplus \mathcal{W} = \mathbb{C}^n$, there are constructive ways to generate the associated unique projector P with range $\mathcal{V}$ and kernel $\mathcal{W}$. If $\dim \mathcal{V} = r$, let $v_1, \ldots, v_r$ be any basis for $\mathcal{V}$, and let $m_1, \ldots, m_{n-r}$ be any basis for $\mathcal{W}^\perp$. Then the projection P is given by

$$P = V \left(W^* V\right)^{-1} W^*, \qquad V = \left[v_1, \ \ldots, \ v_r\right], \qquad W = \left[w_1, \ \ldots, \ w_r\right].$$

Orthogonal projectors in particular can be easily constructed from orthornormal vectors: if $q_1, \ldots, q_r$ are any $r \leq n$ orthonormal vectors in $\mathbb{C}^n$, then the matrix P given by

$$P = QQ^*, \qquad Q = \left[q_1 \ \ q_2 \ \ \cdots \ \ q_r\right] \in \mathbb{C}^{n \times r},$$

is the orthogonal projector onto the span of the q_j.
All projection matrices P satisfy $\|P\|_2 \geq 1$, with equality if and only if P is an orthogonal projector. Orthogonal projection matrices have eigenvalues either 1 or 0.