Analysis of Numerical Methods I
MTH6610 – Section 001 – Fall 2017

Lecture notes
Wednesday August 23, 2017

These notes are not a substitute for class attendance. Their main purpose is to provide a lecture overview summarizing the topics covered.

Reading: Trefethen & Bau III, Lectures 1, 2, 3
Reading: Isaacson & Keller, Sections 1.1

Vectors in $\mathbb{C}^n$ are written with lowercase letters, e.g., v, and matrices in $\mathbb{C}^{m\times n}$ in uppercase letters, e.g., A. We assume you know the following basic concepts and topics:

- matrix-vector and matrix-matrix multiplication
- outer products
- basis, linear independence, rank, matrix inverse, determinant, eigenvalues
- complex arithmetic
- inner products
- matrix (Hermitian) transpose

Given $u, v \in \mathbb{C}^n$, the inner product of two vectors is given by

$$\langle u, v\rangle = v^* u = \sum_{j=1}^{n} \bar{v}_j u_j,$$

where $\bar{v}_j$ denotes the complex conjugate of $v_j \in \mathbb{C}$. Standard linear algebraic operations can be defined using the inner product: angles between vectors, lengths of vectors, scalar and vector projections, orthogonality.

Definition 1
A set V of vectors is an orthogonal set if $\langle u, v\rangle = 0$ for all $u, v \in V$ with $u \neq v$. V is a set of orthonormal vectors if, in addition to being an orthogonal set, each vector in V has unit length.

Orthonormal vectors are useful since they can be used to easily perform orthogonal decompositions of arbitrary vectors.

Definition 2
A matrix $U \in \mathbb{C}^{n\times n}$ is unitary if $U^* = U^{-1}$. (Thus $U^*U = I$, where I is the $n \times n$ identity matrix.)

Unitary matrices preserve lengths and Euclidean structure.
The standard approach to computing the "size" of a vector is a *norm*.

Definition 3
A map $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ is a norm if it satisfies all of the following properties for every $x \in \mathbb{C}^n$ and every $c \in \mathbb{C}$:

a. $\|x\| \geq 0$, *with* $\|x\| = 0$ *iff* $x = 0$
b. $\|x + y\| \leq \|x\| + \|y\|$
c. $\|cx\| = |c| \, \|x\|$

There are many ways of defining norms; the most common are the p-norms:

$$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}, \qquad 1 \leq p < \infty,$$

along with $\|x\|_\infty = \max_{1 \leq j \leq n} |x_j|$. Various relationships between these norms exist, for example, Hölder's inequality:

$$| \langle u, v \rangle | \leq \|u\|_p \, \|v\|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

A special case of this with $p = q = 2$ is the Cauchy-Schwarz inequality.
Norms on matrices are operations $\| \cdot \|$ satisfying the same properties as for vector norms. Two popular matrix norms are the induced (p, q)-matrix norms

$$\|A\|_{p,q} = \sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_p}{\|x\|_q}.$$

When $p = q$ we write $\|A\|_p$. Another popular norm is the Frobenius norm,

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |A_{i,j}|^2 \right)^{1/2} = \sqrt{\text{trace}\,(A^* A)}.$$

Induced matrix norms are *submultiplicative*. The induced 2-norm and the Frobenius norm are invariant under unitary multiplication.