Midterm
MATH6610, Section 01

Name: ______________________________

October 6, 2017

This test is:

- closed-book
- closed-notes
- no-calculator
- 50 minutes

Indicate your answers clearly, and show your work. Partial credit will be awarded based on work shown. Full credit will not be awarded without some work shown.

Fun fact of life: if your work is not legible, I will not be able to read it. The ramifications of this outcome should be clear.

There are 3 questions, some with multiple parts; each question is worth a total of 20 points.

All pages are one-sided. If on any problem you require more space, use the back of the page.

Do not turn this page until directed to begin

1. (20 pts) Using the SVD, prove that any matrix in $A \in \mathbb{C}^{m \times n}$ is the limit of a sequence of matrices of full rank. Use the 2-norm in your proof.

2. a.) (15 pts) Let P be a projection matrix. Prove that $\ker(P) \perp \text{range}(P)$ if and only if $P = P^*$. (You cannot use the definition of an orthogonal projector as a proof.)

b.) (5 pts) Can you design a projection matrix P satisfying

$$\operatorname{range}(P) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}, \qquad \ker(P) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right\} \quad ?$$

If so, explicitly provide such a projection matrix. If it's not possible, explain why not.

3. a.) (15 pts) Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$, and suppose it has the QR decomposition

$$A = QR, \tag{1}$$

with $Q \in \mathbb{C}^{m\times n}$ and $R \in \mathbb{C}^{n\times n}$. Show that A has rank n if and only if all the diagonal entries of R are nonzero.

b.) (5 pts) Let $A \in \mathbb{C}^{m \times n}$ with $m \leq n$. (Note this is the reverse inequality compared to the previous part!) Assume again that A has a QR decomposition as in (1) on the previous page, but now Q is $m \times m$ and R is $m \times n$.

Suppose that the diagonal of R has exactly k nonzero entries, with $0 \leq k \leq m$. With this knowledge, can you provide tight bounds on the rank of A in terms of k, m, and n? Justify your answer.