Department of Mathematics, University of Utah

Analysis of Numerical Methods I

MATH 6610 – Section 001 – Fall 2017

Homework 4

Approximation techniques

Due Friday, December 1, 2017

P1. This problem concerns univariate polynomial interpolation.

(a) Let $f(x) = x^3 - 1$. Without a computer, compute the degree-3 polynomial that interpolates $f(x)$ at $x = -1, 0, 1, 2$.

(b) Let $g(x) = x^4 - 1$. Without a computer, compute the degree-3 polynomial that interpolates $g(x)$ at $x = -1, 0, 1, 2$.

(c) Let $h(x) = 1/(1 + 5x^2)$. Let $h_N(x)$ denote the degree-$(N-1)$ polynomial interpolant of $h(x)$ at N equispaced points on the interval $[-1, 1]$. Write a program that plots h and the interpolant h_N for $N = 5, 10, 20, 30, 50$.

(d) Write a program that plots the Lebesgue function for equispaced points on this interval for $N = 5, 10, 20$. Use this to explain your findings in the previous part.

(e) Let $j_N(x)$ denote the degree-$(N-1)$ polynomial interpolant of $h(x)$ at N Chebyshev points on $[-1, 1]$. Write a program that plots h and the interpolant h_N for $N = 5, 10, 20, 30, 50$.

(f) Write a program that plots the Lebesgue function for Chebyshev points on this interval for $N = 5, 10, 20$. Use this to explain your findings in the previous part.

P2. Let $w(x)$ be a strictly positive, bounded weight function on an interval I on the real line. (I may be unbounded if w decays at infinity sufficiently quickly.) Given $x_1, \ldots, x_N \in I$, let I_N be the associated degree-$(N-1)$ polynomial interpolation operator, i.e., if f is continuous, then $I_N f$ is degree-$(N-1)$ polynomial that interpolates f at the x_j. Define

$$C_w(I) = \left\{ f : I \to \mathbb{R} \;\middle|\; \|f\|_{w,\infty} < \infty \right\}, \qquad \|f\|_{w,\infty} \coloneqq \sup_{x \in I} w(x)|f(x)|.$$

Prove the following weighted version of Lebesgue's Lemma,

$$\|f - I_N f\|_{w,\infty} \leq [1 + \Lambda_w] \inf_{p \in P_{N-1}} \|f - p\|_{w,\infty},$$

where P_{N-1} is the space of polynomials of degree at most $N - 1$, and

$$\Lambda_w = \sup_{x \in I} w(x) \sum_{j=1}^{N} \frac{|\ell_j(x)|}{w(x_j)},$$

where $\ell_j \in P_{N-1}$ is the cardinal Lagrange interpolant, $\ell_j(x_i) = \delta_{i,j}$.

P3. Define the standard L^2 Sobolev spaces of periodic functions on $[0, 2\pi]$: Given a non-negative integer s,

$$H_p^s([0, 2\pi]) = \left\{ f : [0, 2\pi] \to \mathbb{C} \;\middle|\; f^{(r)}(0) = f^{(r)}(2\pi) \text{ for } r = 0, \ldots s-1, \text{ and } \|f\|_{H_p^s} < \infty \right\},$$

where $f^{(r)}$ denotes the rth derivative of f (with $f^{(0)} \equiv f$), and

$$\|f\|_{H_p^s}^2 = \sum_{j=0}^{s} \left\| f^{(j)} \right\|_{L^2}^2 = \sum_{j=0}^{s} \int_0^{2\pi} \left| f^{(j)}(x) \right|^2 \mathrm{d}x.$$

Let f_N denote the frequency-$(N-1)$ Fourier Series approximation to f on $[0, 2\pi]$, i.e.,

$$f_N(x) = \sum_{|j|<N} \widehat{f}_j(x) \frac{1}{\sqrt{2\pi}} e^{ijx}.$$

Prove that,

$$\|f - f_N\|_{H_p^j} \leq N^{j-s} \|f\|_{H_p^s}, \qquad 0 \leq j \leq s$$

P4. This problem concerns interpolative quadrature formulas. All these problems should be done *without* a computer.

(a) Compute weights for the closed 4-point Newton-Cotes rule on $[-1, 1]$.

(b) Consider weights w_j and w'_j for a quadrature rule of the form

$$\int_0^1 f(x)\,\mathrm{d}x \approx w_0 f(0) + w_1 f(1) + w'_0 f'(0) + w'_1 f'(1),$$

where f' is the derivative of f. Compute these weights for a quadrature rule that is exact for all polynomials up to degree 3.

(c) Consider a quadrature rule of the form

$$\int_{-1}^1 f(x)\,\mathrm{d}x \approx \sum_{j=1}^{3} w_j f(x_j),$$

and assume that the x_j are distinct points. Someone claims that this quadrature rule is exact for all polynomials up to degree 3. Is this possible? If so, give conditions on x_j that must be satisfied for this to hold. If it's not possible, prove that it's not possible.

P5. This problem concerns interpolative differentiation formulas. All these problems should be done *without* a computer.

(a) Given $h > 0$, compute weights for the following one-sided differentation formula:

$$f'(x) = w_0 f(x) + w_1 f(x+h) + w_2 f(x+2h) + \mathcal{O}(h^2)$$

(b) Given $h > 0$, compute the weights for the following central differentiation formula:

$$f''(x) = w_{-1} f(x-h) + w_0 f(x) + w_1 f(x+h) + \mathcal{O}(h^2)$$