Department of Mathematics, University of Utah

Analysis of Numerical Methods I

MATH 6610 – Section 001 – Fall 2017

Homework 2

Orthogonalization and the QR decomposition

Due Monday, October 2, 2017

Trefethen & Bau III, Lecture 6: # 6.2, 6.4
Trefethen & Bau III, Lecture 7: # 7.3, 7.5
Trefethen & Bau III, Lecture 9: # 9.2 (You need not internalize lecture #9 to do this problem)
Trefethen & Bau III, Lecture 10: # 10.1, 10.2
Trefethen & Bau III, Lecture 12: # 12.3

Additional problems:

P1. Let $M \in [1, \infty)$ and $n \geq 2$ be arbitrary. Explicitly construct a projection matrix $P \in \mathbb{R}^{n \times n}$ such that $\|P\|_2 = M$.

P2. Let P be a projection matrix. Show that $\ker(P) \perp \operatorname{range}(P)$ if and only if $P = P^*$. (I.e., I am asking you to prove a result that justifies the definition of an orthogonal projector.)

P3. Let $A \in \mathbb{C}^{m \times n}$ have rank r and *reduced* SVD

$$A = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^*$$

The *Moore-Penrose pseudoinverse* of A is defined as

$$A^+ = \widetilde{V}\widetilde{\Sigma}^{-1}\widetilde{U}^*$$

Prove the following:

(a) If A is invertible (hence also square), then $A^{-1} = A^+$.

(b) Prove that the matrices AA^+, A^+A, $I - AA^+$, and $I - A^+A$ are all projection matrices. Are they orthogonal? How would you characterize their ranges and kernels in terms of subspaces defined by the matrix A?

(c) Give and prove conditions on A so that $AA^+A = A$.

(d) Show that $\|A^+\|_2 = 1/\sigma_r(A)$

(e) In the induced matrix ℓ^2 norm $\|\cdot\|$, is the operation $A \mapsto A^+$ well-conditioned? That is, given an arbitrary but fixed A and a perturbation matrix B, is $||(A + B)^+ - A^+||/||A^+||$ controllable by $||B||/||A||$? Prove it, or give a counterexample.

P4. Let $A \in \mathbb{C}^{m \times n}$ and $b \in \mathbb{C}^n$ be given. Show that $x = A^+b$ solves

$$\min_{z \in \mathbb{C}^n} ||z|| \quad \text{subject to} \quad \|Az - b\|_2 \text{ is minimized}$$

What is the difference between this x and the linear least-squares solution to $Az = b$?
