Determination of the Sampling Density Compensation Function Using a Point Spread Function Modeling Approach and Gridding Approximation

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Synopsis

Reconstruction of MRI images from data sampled on arbitrary *k*-space trajectories requires determination of the sampling density compensation function (DCF) that is used to compensate for nonuniform sampling density. We propose a new method for finding the DCF based on a point-spread function (PSF) modeling approach and gridding approximation of the resulting matrix equation. The proposed method for DCF determination was tested on simulated and real MRI data sampled on radial and spiral trajectories and was demonstrated to provide smaller reconstruction errors than other iterative DCF estimation techniques.

Theory and Methods

The PSF of an MRI reconstruction from non-uniformly sampled data weighted by DCF values w_n is defined as $\mathrm{PSF}(\mathbf{r}) = \sum_n w_n \exp(2\pi i \cdot \mathbf{r}\mathbf{k}_n)$. Ideally, the DCF should be chosen in such a way that the resulting PSF approaches the ideal PSF (delta function). In matrix form, the equation modeling this requirement is given by

$$\mathbf{REw} = \mathbf{R\delta}\ , \qquad (1)$$

where **E** is the inverse Fourier transform matrix, **w** is the vector of the DCF values, $\boldsymbol{\delta}$ is vector describing the ideal PSF, and **R** is a diagonal matrix of weights on error in the PSF. The least mean-squares estimation of DCF that minimizes the weighted error in the PSF could be found by solving the system of the normal equations:

$$(\mathbf{E}^H\,\mathbf{R}^H\mathbf{RE})\mathbf{w} = \mathbf{u}\ , \qquad (2)$$

where **u** is the unity vector. To achieve this goal iterative technique minimizing the corresponding quadratic form could be applied. The main problem with such an approach is the large cost of the residual calculation. To significantly decrease memory and computation time requirements, we propose to use a gridding approximation [1] for the system of equations (Eq. 2):

$$(\mathbf{G}_{\mathbf{CA}}\mathbf{F}\mathbf{C}^{-1}\mathbf{R}^{\mathbf{H}}\mathbf{R}\mathbf{C}^{-1}\mathbf{F}^{-1}\mathbf{G}_{\mathbf{AC}})\mathbf{w} = \mathbf{u}\ , \qquad (3)$$

where $\mathbf{G_{CA}}$ and $\mathbf{G_{AC}}$ are matrices representing the convolution operations with gridding kernel c(**k**) from sufficiently dense Cartesian grid to arbitrary k-space positions and vise versa, **C** is the diagonal matrix containing values of C(**r**), image domain representation of c(**k**), **F** and $\mathbf{F^{-1}}$ are direct and inverse Fourier transform matrices. Equation (3) is considerably simplified by choosing **R=C**:

$$(\mathbf{G}_{\mathbf{CA}}\mathbf{G}_{\mathbf{AC}})\mathbf{w} = \mathbf{u}\ . \qquad (4)$$

Such a choice of **R** results in weighting the error in the PSF with C(**r**). Equation (4) is further simplified by substituting two interpolation operations by one convolution with the corresponding kernel:

$$\mathbf{M}\mathbf{G}_{\mathbf{AA}}\mathbf{w} = \mathbf{M}\mathbf{u}\ , \qquad (5)$$

where $\mathbf{G_{AA}}$ is the matrix representing the convolution among the positions of the k-space trajectory and **M** is the diagonal matrix that is used to weight the contributions of different k-space sampling positions. It should be noted that the result obtained here was derived using the criterion on target PSF introduced in [2].

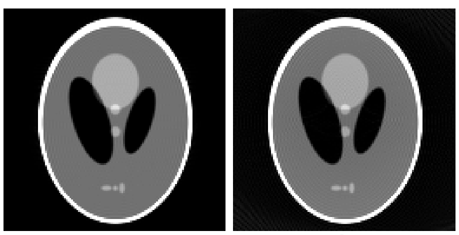
The solution of Eq. (5) may be unstable because of ill conditioning of $\mathbf{G_{AA}}$. We developed a special algorithm to obtain the stable solution of the system. In its general form, the algorithm combines the optimal iterations of a steepest descent method with projection onto a convex set representing all non-negative solutions. Starting with the DCF of Jackson [3], $\mathbf{w}_{(0)} = (\mathbf{G}_{\mathbf{AA}}\mathbf{u})^{-1}$, each next estimate is obtained as

$$\mathbf{w}_{(n+1)} = P\mathbf{w}_{(n)} + \alpha_{(n)}\mathbf{r}_{(n)}\ , \qquad (6)$$

where

$$\alpha_{(n)} = \frac{\mathbf{r}_{(n)}^T\mathbf{r}_{(n)}}{\mathbf{r}_{(n)}^T\mathbf{G}_{\mathbf{AA}}\mathbf{r}_{(n)}},\ \mathbf{r}_{(n)} = \mathbf{M}\left(\mathbf{u} - \mathbf{G}_{\mathbf{AA}}\mathbf{w}_{(n)}\right),\ Pw_i = \begin{cases} w_i,\ w_i > 0 \\ 0,\ \textit{otherwise} \end{cases}.$$

Additional stability is gained by choosing **M** to contain Jackson's DCF values, i.e. $\mathbf{M} = diag(\mathbf{w}_{(0)})$.



a b

Figure 1. Shepp-Logan image reconstructed from data sampled on regular Cartesian grid (a) and spiral trajectories (b) with the DCF estimated by the method presented in this paper (50 iterations).

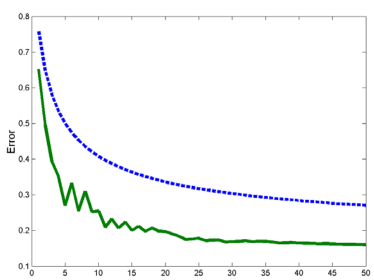


Figure 2. RMS error of the difference image between the reference image and images from spiral data reconstructed with Pipe's DCF (dotted line) and with the DCF estimated by the method proposed in this paper (solid line). The error is measured in units of the reconstruction error for Jackson's DCF.

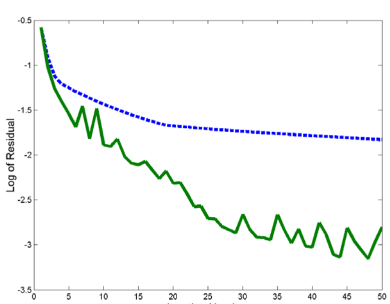


Figure 3. Logarithm of L_∞ norm of residual vs. the number of iterations for Pipe's method (dotted line) and the presented method (solid line)

Results

The computer-generated data were used for comparison of the proposed DCF estimation method with existing techniques for the DCF calculation [2,3]. The spiral and Cartesian grid data were calculated using an analytical expression for the *k*-space representation of Shepp-Logan phantom and then multiplied by Gaussian function to decrease Gibbs artifact. The spiral data consisted of 13 interleaves, 1800 data points per interleave and were calculated for matrix size 128×128. The Kaiser-Bessel convolution kernel (L=3, B=13.9086) was employed in both method of Pipe [2] and the presented method as well as for gridding reconstruction. Non-negativity constraint was used to stabilize the solution of Eq. (5). The image reconstructed from Cartesian data was considered as a reference image. In order to eliminate the influence of different DCFs, all images were normalized to have the same mean intensity as the mean value of the reference image. Both RMS (Fig. 2) and L_∞ norm of residual plots (Fig. 3) demonstrate that our new method provides much smaller reconstruction error and the solution is achieved in fewer iterations.

Discussion

Our results demonstrate that the new method for estimating the DCF produces weights that provide significantly less reconstruction error in comparison with the existing techniques [2,3]. It can be shown that only one regridding per iteration is required if no nonlinear projection is used, and two such operations otherwise. The primary necessity of non-negativity constraint is to increase the stability of the solution. From our experience, if small (up to 50) number iterations are used, the omission of non-negativity constrain is safe and typically results in even smaller reconstruction errors. However, inclusion of non-negativity constraint is required to ensure the algorithm stability for arbitrary number of iterations. The matrix formulation of the DCF estimation problem presented in this paper allows for further investigation of fast, efficient and stable algorithms for retrieval of DCF using other well-understood techniques such as conjugate gradients. Further research is required on the dependence of the reconstruction error on the convolution kernel characteristics.

Acknowledgements

This work was supported in part by NIH BISTI grant 1P20HL68566-01 and NIH grant R01HL48223.

References:[1] Pruessmann KP, et al, MRM 2001;46:638–651; [2] Pipe JG, et al, MRM 1999;41:179-186; [3] Jackson JI, et al, IEEE TMI 1991;MI-10:473-478.