Mesh Quality: A Function of Geometry, Error Estimates or Both?

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Abstract. *The issue of mesh quality for unstructured triangular and tetrahedral meshes is considered. The theoretical background to finite element methods is used to understand the basis of present-day geometrical mesh quality indicators. A survey of more recent work in the development of finite element methods reveals work on anisotropic meshing algorithms and on providing good error estimates that reveal the relationship between the error and both the mesh and the solution gradients. The realities of solving complex three dimensional problems is that such indicators are presently not available for many problems of interest. A simple tetrahedral mesh quality measure using both geometrical and solution information will be described. Some of the issues in mesh quality for unstructured tetrahedral meshes will be illustrated by means of a simple example.*

keywords. Mesh quality, unstructured meshes, error estimates, mesh generation.

1 Introduction

The range of problems solved by finite element and finite volume p.d.e. solvers based on triangular and tetrahedral meshes e.g [7] [46] is rapidly increasing. The original applications problem class for many such solvers was in the area of solid mechanics and elasticity in particular. These methods are being applied at present to a wide range of problems in solid and fluid mechanics ranging from linear elasticity to turbulent flows, [24]. This very broad spectrum of applications naturally raises the issue of whether or not the meshes being used are appropriate for the applications being considered.

The issue of whether the mesh is appropriate to represent the solution has been investigated almost as long as finite elements have been used. In order to state the important finite element results that formed a basis for existing mesh quality measures it is necessary to introduce some notation. Without loss of generality the case of linear finite elements on triangular or tetrahedral meshes will be considered. Define the error as being the difference between the linear approximation, u_{lin} and the true solution u i.e. $e_{lin}(x,y) = u_{lin}(x,y) \; - \; u(x,y)$. The L^2 error norm is defined by $||e_{lin}(x,y)||_{L^2}$ where

$$||e_{lin}(x,y)||^2_{L^2} = \int_T (e_{lin}(x,y))^2 dxdy \; . \tag{1}$$

The H^1 error norm is defined by $||e_{lin}(x,y)||_{H^1}$ where

$$||e_{lin}(x,y)||^2_{H^1} = \int_T (e_{lin}(x,y))^2 + (e_{lin,x}(x,y))^2 + (e_{lin,y}(x,y))^2 dxdy \; . \tag{2}$$

The seminorm of the H^2 space is defined by $|u|_2$ where

$$|u|_2 = \left(\sum_{|\delta|=2} \frac{2!}{\delta_1 ! \delta_2 !} ||(\partial_x)^{\delta_1} (\partial_y)^{\delta_2} u||^2_{L^2} \right)^{1/2} \; . \tag{3}$$

Aside from the notion that meshes with regular or smoothly varying element sizes are more aesthetically pleasing, the starting point for the notion of mesh quality would appear to be the analysis leading to the minimum angle condition that the smallest angle should be bounded away from zero. This perhaps originated with Zlamal [47] and is quoted by Strang and Fix [40] together with a statement regarding how "poorly shaped" triangles may have an effect on the condition number of the linear algebra problem that must be solved. The correct version of this result came with the analysis of Babuska and Aziz [5], who showed that the requirement for triangles was that there should be no large angles. The general results of both Zlamal and Babuska and Aziz are of the form

$$||e_{lin}(x,y)||^2_{H^1} \;\; \leq \;\; \Gamma(\theta) |u|_2 \tag{4}$$

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where Zlamal [47] showed that $\Gamma(\theta) = h/sin(\theta_{min})$ for the minimum angle $\theta_{min} = min(\theta_1, \theta_2, \theta_3)$, see Figure 1. In contrast Babuska and Aziz showed that $\Gamma(\theta) = h/\Psi(\theta)$ where $\Psi(\theta)$ is a positive continuous and finite function and for $\theta \leq \gamma < \pi$, $\Psi(\theta) \geq \Psi(\gamma)$ where γ is a bound on the maximum interior angle of the triangle in Figure 1. This work was extended, much later, to tetrahedral elements by Krizek [26] in a similar spirit.

The precise way that these results influenced mesh generation code writers is unclear. Early mesh generation papers are covered by the surveys of Shephard [38] and Thacker [41]. In these surveys there is little explicit reference to how the theoretical work has been adopted, though Thacker does say that elements should be nearly equilateral otherwise instability may result. More recent surveys by Bern and Epstein [12] and Nielson [33], do mention the theoretical results and the monographs of Carey [16] and George and Borouchaki [20] treat the subject in more detail. The perceived meshing wisdom has thus been that if possible elements should have no small or large angles. In the case of tetrahedral meshes this has has led to geometric mesh quality indicators as described in Liu and Joe [28]. One example being Weatherill's edge quality estimator for tetrahedra of volume V and edge lengths h_i:

$$Q_w = \frac{1}{8.48528V} \left[(\Sigma \frac{h_i}{6})^3 \right] \quad . \tag{5}$$

Such indicators do a good job of identifying geometric imperfections in the mesh -an important task before any solution is computed on the mesh. The difficulty is that it is unclear that such indicators are valid for every solution on every mesh. The ideal solution is thus to understand the relationship between the error and the mesh. Recently there have been many attempts to dynamically modify triangular meshes so as to fit the solution better. Some of these methods will be described below - most of them lead to stretched meshes for anisotropic solutions. The main requirement is thus for error estimators that include both solution and geometry information. Such estimators are still in their infancy especially in 3D but it will be shown that it is possible to use interpolation errors, [13] and through a simple example on a tetrahedral mesh that the accuracy in the solution can depend critically on the mesh.

2 A Quality Indicator Based upon Finite Element Interpolation Theory

The decision as to whether or not (and how) a mesh should be refined should be based on an error estimate that reflects not only the interpolation error caused by approximating the solution by a finite element space on a given mesh but also the discretization error of the numerical method used to approximate the p.d.e. and the choice of norm used to measure the error. Rippa [36] makes a convincing case based on interpolation errors that long thin triangles do indeed form part of a good mesh for strongly anisotropic solutions. A good discussion of this topic also occurs in Nielson [33].

Berzins [13] derives a new mesh quality indicator from the work of Nadler [31] which gives a particularly appropriate expression for the interpolation error when a quadratic function is approximated by a piecewise linear function on a triangle. Consider the triangle T defined by the vertices v_1, v_2 and v_3 as shown in Figure 1. Let h_i be the length of the edge connecting v_i and v_{i+1} where $v_4 = v_1$. Nadler [31] considers the case in which a quadratic function

$$u(x,y) \;=\; \frac{1}{2}\underline{x}^T \; H \; \underline{x} \; where \quad \underline{x} \;=\; [x,y]^T \;, \tag{6}$$

where H is a constant $2x2$ real matrix, is approximated by a linear function $u_{lin}(x,y)$, as defined by linear interpolation based on the values of u at the vertices and shows that the error denoted by equation (1) above satisfies

$$\int_T (e_{lin}(x,y))^2 dx \; dy \;=\; \frac{A}{180} \left[((d_1 + d_2 + d_3)^2 + {d_1}^2 + d_2^2 + d_3^3 \right] \tag{7}$$

where A is the area of the triangle and $d_i = \frac{1}{2}(v_{i+1} - v_i)^T \; H \; (v_{i+1} - v_i)$ is the edge derivative along the v_i and v_{i+1} edge. Berzins [12] uses this result as the basis for an indicator that takes into account both the geometry and the solution behaviour by defining scaled edge derivatives by $\tilde{d}_i = |d_i|/d_{max}$ where $d_{max} \;=\; max\left[|d_1|, |d_2|, |d_3|\right]$. For notational convenience define $\underline{\tilde{d}} = [\tilde{d}_1, \tilde{d}_2, \tilde{d}_3]^T$ and

$$\tilde{q}(\underline{\tilde{d}}) \;=\; (\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3)^2 + \tilde{d}_1^2 + \tilde{d}_2^2 + \tilde{d}_3^2 \tag{8}$$

A measure of the anisotropy in the derivative contributions to the error is then provided by

$$q_{aniso} \;=\; \tilde{q}(\underline{\tilde{d}})/12 \;. \tag{9}$$

The relationship between q_{aniso} and the linear interpolation error is that in the case when the matrix H is positive definite, i.e. $d_i > 0$, then the indicator q_{aniso} is a scaled form of the interpolation error, [13], in this special case.

A consistent and related but geometry-only based indicator is then defined by:

$$q_m(\underline{h}) = \tilde{q}(\underline{h})/(16\ \sqrt{3}\ A), \quad where\ \underline{h} = [h_1, h_2, h_3]^T, \tag{10}$$

has value 1 for an equilateral triangle and tends to the value infinity as the area of a triangle tends to zero but at least one of its sides is constant. Bank [7] and Weatherill's [46] indicators are denoted by q_b and q_w and defined by

$$\frac{1}{q_b} \ = \ \frac{1}{4\ \sqrt{3}\ A} \left[(h_1^2 + h_2^2 + h_3^2 \right], \quad q_w \ = \ \frac{1}{3\ A} \left[(h_1 + h_2 + h_3)^2 \right] \tag{11}$$

respectively. Hence, from equations (8) and (9) the connection between these indicators is that

$$q_m(\underline{h}) = \ \frac{1}{4\ q_b} \ + \ q_w\ \frac{\sqrt{3}}{16}\ . \tag{12}$$

The choice of norm is not often considered but may be critical in deciding what is the best mesh. Given the linear interpolation error defined by equation (2), Berzins [12] considers the example of Babuska and Aziz [5] in which triangles of the form of that in Figure 1 are used to interpolate the function x^2 with x horizontal. Berzins [14] shows that in the L_2 norm the isosceles triangle is more accurate whereas in the H^1 norm right triangles are more accurate and the isosceles triangle is the worst choice as $\alpha \downarrow 0$ in Figure 1. Hence a good mesh in one norm is not a good mesh in another norm.

The extension to the case of non-quadratic functions may be considered by assuming that the exact solution is locally quadratic. Bank [7] uses such an approach inside the code PLTMG and calculates estimates of second derivatives. Adjerid, Babuska and Flaherty [1] use a similar approach based on derivative jumps across edges to estimate the error. An alternative approach is to use the ideas of Hlavacek et al. [21] to estimate nodal derivatives and hence second derivatives.

3 Mesh Movement Redistribution in 2/3D

The idea that it is important for the the shape of the elements to reflect local solution behaviour, particularly for highly directional flow problems, is well-known [15, 9, 27]. One of the significant steps in realising this understanding was the Moving Finite Element method of Keith Miller, see Baines [6], which continuously moves the mesh for transient problems. Some of the meshes shown by Baines are highly distorted. A similar approach, but rather simpler, was derived by Peraire et al. [34], who applied a simple local iterative procedure based on quantities such as pressure gradients to produce stretched meshes for highly-directional Euler equations flow problems. A key part of their algorithm is a simple Laplacian smoothing approach that has also been used by many others, e.g. Barth [9, 10].

A slightly different approach still is employed by Tourigny and Baines [42], who investigate the construction of locally optimal piecewise polynomial fits to data and produce meshes which vary from smooth to skewed, depending on the solution. The idea is further extended by Tourigny and Hulseman [43], who minimise an energy functional using a Gauss-Siedel method locally to get similarly skewed meshes.

Beinert and Kroner [11] move edges so that they are aligned with shock waves and also define a *Blue* directional refinement approach. For example in the right side of Figure 1 if the edges eT_1, eT_2 are parallel and aligned with the flow direction then the pairs of triangles is replaced by four anisotropic triangles. Although the indicator used to guide refinement is the gradient of the Mach number rather than an explicit error estimator, the results are nevertheless impressive.

The relative size of the edge indicators, d_i defined by equation(7) in the previous section gives a means of indicating which edges should be refined to reduce the error. One recent method to take advantage of such local gradients is the modified Delaunay approach of Borouchaki et al. [15] in which the local gradient information, of the form of d_i values, is used in conjunction with the Delaunay mesh generator to compute highly stretched grids for anisotropic flows in two space dimensions. The results presented by Borouchaki et al. show that this approach can give good results on problems with highly directional flows.

Other methods using the gradient quantities d_i defined in the previous section are the mesh generation procedure of Simpson [37] and the mesh modification procedure of Ait-Ali-Yahia et al.[2]. In the latter case the H matrix is

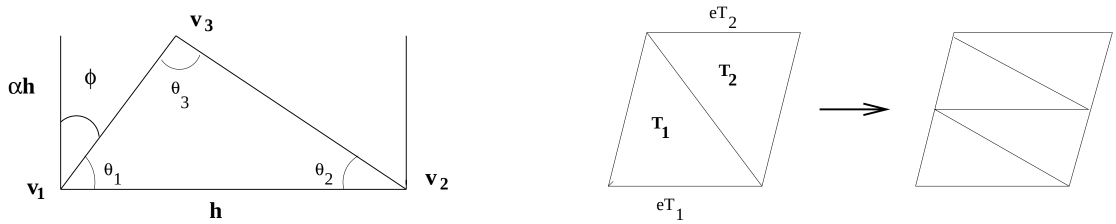

Figure 1: Babuska and Aziz Example Triangle and Blue Refinement of Two Triangles into Four.

modified to be positive definite and edge indicators, defined in the notation used here by $d_i/\sqrt{\Delta x_i^2 + \Delta y_i^2}$, are used to move the mesh. This approach thus scales the edge error component by the edge length. Ait-Ali-Yahia et al. [2] interpret d_i as the edge length in the H norm.

Mesh redistribution in 3D is less common but Freitag and Ollivier-Gooch [19] and Iliescu [23] give interesting algorithms for splitting tetrahedra. In Iliescu's approach pairs of tetrahedra satisfying convexity and angle conditions related to the flow direction are split into three tetrahedra so as to be aligned with the flow direction, see Figure (2). Freitag and Ollivier-Gooch [19] also provide convincing evidence that mesh smoothing can have beneficial consequences for the rate of convergence of the iterative solver.

A common feature of all the methods listed in this section is that although the mesh is improved in some sense, the criterion used is only indirectly related to the error.

4 Error estimators with Geometry effects

Recent work in error estimates is starting to reveal the explicit dependence of the error on both solution derivatives and on the mesh. An important stepping stone in this process was the work of Appel, [3, 4], which proved that one can benefit from the presence of small and even large angles of the elements. Appel also shows for bilinear elements that the interpolation and finite element errors coincide. Tsukerman [44, 45] derives a maximal eigenvalue condition which shows that it is the maximum eigenvalue of the element stiff matrix that characterises the impact of the shape of the element on the energy norm of the error of the finite element approximation.

Bank and Smith [7], in error analysis for the method used in the PLTMG code shows how the error can be written using d_i and q_b from Section 2 as a quotient of solution and geometry information:

$$\int_t |\bigtriangledown e_{lin}(x,y)|dx \approx \frac{d_1^2 + d_2^2 + d_3^2}{q_b} \tag{13}$$

This somewhat simpler form than the expressions in equation(7) and [14] comes about because Bank and Smith consider only the diagonal terms in a matrix to arrive at their approximation. While this error estimator only applies to steady problems Lang [27] considers transient problems and explicitly includes both solution derivative and geometry information in the error estimates he derives. For 2D reaction-diffusion p.d.e.s modelling highly-directional phenomena such as flame propagation, Lang proves the error estimate:

$$||e_{lin}(x,y)||^2_{H^1} \quad \leq \quad \tilde{c}\left(\sum_{T \epsilon T_k} \eta_T^2\right)^{1/2} \tag{14}$$

where the local error estimator $\eta_T^2 = C^2(\tau,\lambda,T), D_T^2 U$ and $D_T^2 U$ is a computed approximation to $|u|_2$ as defined by equation (3). The constant $C(\tau,\lambda,T)$ is defined by

$$C(\tau,\lambda,T) = (1+|\lambda|+\lambda^2)^2 h^2 (0.2587(1+\frac{1}{\tau})h^2 + \frac{1}{\pi^2}(1+|\lambda|+\lambda^2)) \tag{15}$$

and where with reference to Figure 1, $\lambda = tan(\phi)$, h is the longest edge and τ is the timestep. This estimate thus precisely describes the effect of both the geometry and the solution on the error and enables decisions regarding directional refinement to be taken.

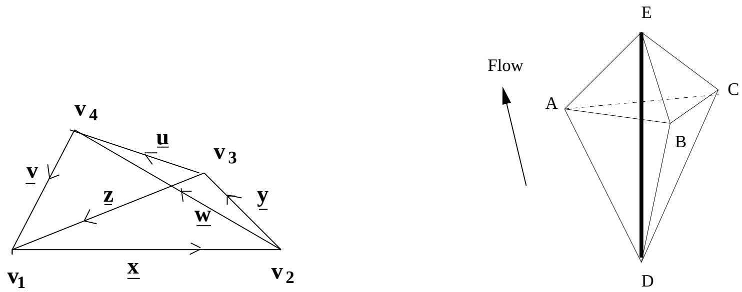

Figure 2: Example Tetrahedron and Iliescu's Directional Refinement Procedure.

5 Linear Tetrahedral Approximation of a Quadratic Function

Although there are now data-dependent tetrahedralisations, see Nielson [33], there are unfortunately very few error estimates for tetrahedral meshes that show the explicit dependence of the error on the mesh and the solution. The natural starting point is perhaps to try and use the interpolation error to assess how appropriate the mesh is for the computed solution. The simple mesh quality indicator of Berzins [13, 14] is based on linear interpolation error estimates and is derived by extending Nadler's [31] approach to tetrahedra by considering the case in which a quadratic function

$$u(x,y,z) \;=\; \frac{1}{2}\, \underline{x}^T \; H \; \underline{x} \; where \quad \underline{x} \;=\; [x,y,z]^T \tag{16}$$

is approximated by a linear function $u_{lin}(x,y,z)$ defined by linear interpolation based on the values of u at the vertices of a tetrahedron T defined by the vertices v_1, v_2 , v_3 and v_4 as shown in Figure 2.

Let h_i be the length of the edge connecting v_i and v_{i+1} where $v_5 = v_1$. With reference to Figure 2 define the vectors $\underline{\hat{x}}$, $\underline{\hat{y}}$, $\underline{\hat{z}}$, $\underline{\hat{u}}$, $\underline{\hat{v}}$ and $\underline{\hat{w}}$ by $v_2 = v_1 + \underline{\hat{x}}$, $v_3 = v_2 + \underline{\hat{y}}$, $v_1 = v_3 + \underline{\hat{z}} v_4 = v_1 - \underline{\hat{v}}$, $v_4 = v_2 + \underline{\hat{w}}$, $v_4 = v_3 + \underline{\hat{u}}$. Berzins [13] defines the vector of second directional derivatives along edges by

$$\underline{d}^T = \frac{1}{2}\, [d_1, ..., d_6\,] \;=\; \frac{1}{2}\, \left[\underline{\hat{x}}^T H \underline{\hat{x}},\;\; \underline{\hat{y}}^T H \underline{\hat{y}},\;\; \underline{\hat{z}}^T H \underline{\hat{z}},\;\; \underline{\hat{u}}^T H \underline{\hat{u}},\;\; \underline{\hat{v}}^T H \underline{\hat{v}},\;\; \underline{\hat{w}}^T H \underline{\hat{w}}\,\right].$$

and shows that the error may written in terms of the six directional derivatives along the edges d_i as:

$$\int_T (e_{lin}(x,y,z))^2 dx\; dy\; dz = \;\frac{6}{4}\; V\; \frac{8}{7!}\;\; \left[(\Sigma d_i)^2 - d_1 d_4 - d_2 d_5 - d_3 d_6 + \Sigma d_i^2\;\right] \quad . \tag{17}$$

It is then possible to define the mesh quality indicator in the same way as in Section 2 in that the error is scaled by the maximum directional derivative d_{max}, the integral is scaled by the volume before taking the square root. In a similar way to as in Section 2 define

$$\tilde{Q}(\underline{\tilde{d}}) \;=\; \left[(\Sigma \tilde{d}_i)^2 - \tilde{d}_1 \tilde{d}_4 - \tilde{d}_2 \tilde{d}_5 - \tilde{d}_3 \tilde{d}_6 + \Sigma \tilde{d}_i^{\,2}\,\right] \;\; where\;\; \underline{\tilde{d}} = [\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4, \tilde{d}_5, \tilde{d}_6]^T. \tag{18}$$

A measure of the anisotropy in the derivative contributions to the error is then provided by Q_{aniso} and a related geometry based indicator by Q_m where

$$Q_{aniso} \;=\; \tilde{Q}(\underline{\tilde{d}})/39 \;\; and\; Q_m(\underline{h}) = \frac{C}{V}\;\; \left[\,\tilde{Q}(\underline{\tilde{h}})\right]^{\frac{3}{2}} \tag{19}$$

where C is a scaling factor to ensure that the indicator has value one when $h_i = h$. A comparison between this geometry indicator, $Q_m(\underline{h})$, with that of Weatherill Q_w as defined by equation(5) was done by Berzins [13] who showed that the values of the two indicators are very similar. The anisotropic interpolation example used by Berzins, [14], shows that in such circumstances it is important to use indicators such as Q_{aniso} which involve solution information.

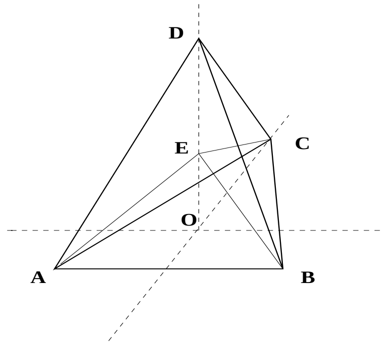

Figure 3: Example Mesh of Four Tetrahedra: ABCE,ABED,ACED and BCED

6 Example Laplace's Equation with Anisotropic Tetrahedra and Finite Element/Volume Schemes

The issue of mesh suitability for a given solution and numerical solver is recognised as a complex one with no easy answers. There are a variety of views concerning the sensitivity of numerical schemes to distorted meshes. Shephard [39] states that the stabilized FEM for example, appear to have no real problem with elements with angles of 179 degrees and 1,000,000 to 1 aspect ratios and that tetrahedra with small angles are well-understood to be needed for boundary layer calculations. In contrast, Millar [29, 30], et al. state that for Laplace's equation, finite volume schemes are less sensitive than finite element schemes to sliver-type tetrahedra in meshes. Given the similarity between the finite volume and element schemes in this case, see [9] the difference may be due to implementation issues such as those discussed by Putti and Cordes [35].

In order to understand better the dependency between the mesh and the error, the Laplaces equation, $\nabla^2 U = 0$, in three space dimensions of [29] will be used. The mesh of five points consists of a single tetrahedrob sub-divided into four by the addition of an internal point and is shown in Figure 3. The analytic solution given by

$$u(x, y, z) = e^{\pi z} cos(\pi y/\sqrt{2}) sin(\pi(x + 0.5)/\sqrt{2}) \tag{20}$$

$$O = [0, 0, 0]^T \ , \ A = [-0.5, -0.5, 0]^T, \ B = [0.5, -0.5, 0]^T \ C = [0, 1, 0]^T \ , \ D = [0, 0, 1]^T, and \ \ E = [0, 0, \epsilon]^T$$

where ϵ is a parameter that will be varied to test the sensitivity of the numerical solution to the mesh and in particular to distorted elements. The values at A, B, C, D are given by the exact solution and denoted by U_A, U_B, U_C, U_D. The scheme used to approximate the Laplacian is Barth's cell-vertex scheme [9, 10]. This gives a challenging situation for mesh quality indicators as the region associated with each node is composed of parts of all neighbouring tetrahedra. At point E the Laplacian is approximated by

$$\nabla^2 U = W_{EA}(U_A - u_E) + W_{EB}(U_B - u_E) + W_{ED}(U_A - u_E) + W_{ED}(U_D - u_E) \tag{21}$$

where u_E is the numerical approximation to the exact value U_E and is explicitly defined by the equation

$$u_E = (\ W_{EA}U_A + W_{EB}U_B + W_{EC}U_C + W_{ED}U_D \ )/(W_{EA} + W_{EB} + W_{EC} + W_{ED}) \tag{22}$$

In order that the solution satisfies a maximum principle all the weights W_{**} must be positive. [9, 10]. Barth also shows how this condition may not be met on a distorted mesh, but Putti and Cordes [35] show how to modify the method to avoid this and that this also improves the accuracy.

Denote the exact solution of the problem at node E by U_E then the p.d.e. truncation error, T.Error, is defined by

$$TError = W_{EA}(U_A - U_E) + W_{EB}(U_B - U_E) + W_{ED}(U_A - U_E) + W_{ED}(U_D - U_E) \tag{23}$$

and the relationship between the truncation error and the error is

$$Error = U_E - u_E = -TError/(W_{EA} + W_{EB} + W_{EC} + W_{ED}) \tag{24}$$

Table 1 (Note this is revised from that in the proceedings of the Roundtable) shows the different mesh quality indicators and the interpolation error as the value of ϵ changes for two tetrahedra given by the points ABCE and ACED. The values for the tetrahedra ABED and BCED being similar to those of ACED. With reference to Table 1 Interp is the square of the interpolation error based on the exact solution. Error and T.Error are the error and truncation error defined by equations (23) and (24) respectively. The results in Table 1 show that the anisotropy indicator follows (not surprisingly) the trend of the interpolation error, but that the pointwise discretization error

Table 1: Q_{aniso} , Standard Mesh Quality Q_w and Error Values

	Tet. ABCE			Tet. ACED			Numerical Error		
ϵ	Q_{aniso}	Q_w	Interp	Q_{aniso}	Q_{nw}	Interp	U_E	Err	T. Err
0.001	0.35	621	3.4e-6	0.15	2.2	1.0e-3	-2.6e-2	0.42	-107.
0.01	0.35	62	3.4e-5	0.15	2.2	1.0e-3	-1.7e-2	0.41	-11.4
0.5	0.38	1.5	1.6e-3	0.17	3.9	6.2e-4	5.2e-1	0.01	-0.65
0.99	0.21	1.1	3.6e-3	0.22	211	2.0e-5	1.07	3.2e-3	-0.07
0.999	0.20	1.1	3.6e-3	0.23	211	2.1e-6	1.08	2.8e-5	-0.06

Table 2: Values of the coefficients $W_{ea}, W_{eb}, W_{ec}, W_{ed}$

ϵ	W_{ea}, W_{eb}, W_{ec}	W_{ed}
0.001	8.0e+1	2.52e-1
0.01	9.0	2.72e-1
0.5	8.3e-1	2.5
0.99	7.5e-1	2.2e+2
0.999	7.5e-1	2.4e+3

behaves very differently, especially for small values of ϵ. The low values of the anisotropy indicator Q_{aniso} indicate potential problems. The geometry indicator does a good job of picking up the very large error for small ϵ but also erroneously identifies a problem with ϵ close to one, when the error is small.

The interesting result is that both mesh quality indicators do not really identify the relationship between the mesh and the error in the numerical solution. It is the differing size of the truncation error as caused by the method coefficients that has a dramatic effect on the error. In the case when $\epsilon = 0.001$ the large size of the coefficient W_{ea} and similarly W_{eb}, W_{ec} arises because the face angle between faces such as EBC and ABC is very close to π . Hence in this case the value $U_D$play little part in determining u_E. In contrast when ϵ is close to one only one coefficient is large and u_E is determined almost solely by U_D its closest neighbour. The values of these coefficients are shown in Table 2, the negative values indicating that the mesh is not a good one from the point of view of approximating the diffusion operator, [9].

7 Conclusions

The overall conclusion is that the only really satisfactory approach would seem to be to have an error estimator based on both solution and geometry information This would appear to be true for strongly directional fluid flows for which highly distorted meshes appear to be very effective. One approach to resolving this issue is to have computable error estimates for each solution component. At present, it is still often the case that such estimates may not be available or may not be reliable. It is also the case that the availability of such error estimates will always lag behind the problems being solved by practitioners. Hence the requirement must be to allow the user to supply mesh quality measures and to choose anisotropic remeshing options. There are, of course, many applications areas in which it is still rather difficult to even understand what constitutes a good mesh. One such area is turbulent combustion which may involve the interaction between many chemical species and complex fluid flows. Such problems are like to provide interesting challenges to the meshing community for some time to come.

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