

# TIME STEPPING WITH SPACE AND TIME ERRORS AND STABILITY OF THE MATERIAL POINT METHOD

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**Abstract.** The choice of the time step for the Material Point Method (MPM) is often addressed by using a simple stability criterion, such as the speed of sound or a CFL condition. Recently there have been several advances in understanding the stability of MPM. These range from non-linear stability analysis, through to Von Neumann type approaches. While in many instances this works well it is important to understand how this relates to the overall errors present in the method. Although it has been observed that spatial errors may dominate temporal ones at stable time steps, recent work has made more precise the sources and forms of the different MPM errors. This now makes it possible to understand how the different errors and the stability analysis are connected. At the same time this also requires simple computable estimates of the different errors in the material point method. The use of simple estimates of these errors makes it possible to connect some of the errors introduced with the stability criteria used. A number of simple computational experiments are used to illustrate the theoretical results.

## 1 INTRODUCTION

The Material Point Method (MPM) has proved to be invaluable for many very challenging problems. However in many ways the method is not as far advanced as, say, finite element methods, in terms of accuracy and stability analysis and computable error estimates for errors in both space time. For example the time step in the MPM is often controlled by a simple speed of sound or CFL-type condition. It is also observed that that the spatial error dominates for cases in which the calculation is stable [12]. This raises the question of the precise relationship between the spatial error and stability. This question may be addressed by characterizing the spatial error in the different parts of MPM, estimating this error and showing how parts of this error may be viewed as being part of a speed of sound stability condition. Section 2 describes the MPM method and its application to a simple example problem. A full description of the method and its errors is provided in Section 3. Simple computational estimates of the different mapping errors in MPM are derived in Section 4. In Section 5 these estimates are applied to a

simple model problem solution that shows how well they may work. A simple derivation of the speed of sound stability condition is derived in Section 6. Finally in Section 7 the stability and error conditions are connected.

## 2 MPM MODEL PROBLEM AND METHOD

The description of MPM used here follows [5] in that the model problem used here is a pair of equations connecting velocity  $v$ , displacement  $u$  and density  $\rho$ :

$$\frac{Du}{Dt} = v, \tag{1}$$

$$\rho \frac{Dv}{Dt} = \frac{\partial \sigma}{\partial x} + b(x, t), \tag{2}$$

with a linear stress model  $\sigma = E \frac{\partial u}{\partial x}$  for which Young's modulus,  $E$ , is constant, a body force  $b$ , which is initially assumed to be zero, and with appropriate boundary and initial conditions. For convenience a mesh of equally spaced  $N + 1$  fixed nodes  $X_i$  with intervals  $I_i = [X_i, X_{i+1}]$ , on the interval  $[a, b]$  is used where

$$a = X_0 < X_1 < \dots < X_N = b, \tag{3}$$

$$h = X_i - X_{i-1}. \tag{4}$$

These fixed nodes are referred to as the  $i$  points. It will also be assumed that periodic boundary conditions exist in that

$$\sigma(a)v(a) = \sigma(b)v(b) \tag{5}$$

together with appropriate initial conditions. While the analysis of MPM for time integration error and energy conservation uses the model problem above it does apply more generally and in multiple space dimensions with a few obvious modifications. The computed solution at the  $p$ th particles will be written as  $u_p^n = u(x_p^n, t^n)$ . Suppose that there are  $np$  particles in total. The calculation of the internal forces in MPM at the nodes requires the calculation of the volume integral of the divergence of the stress [14] using

$$f_i^{int} = - \sum_p D_{pi}(x_p^n) \sigma_p V_p \tag{6}$$

The subscript  $pi$  represents a mapping from particles  $p$  to node  $i$  while the subscript  $ip$  represents a mapping from nodes  $i$  to particles  $p$ . The negative sign arises as a result of using integration by parts [5]. The mass at node  $i$  is defined by

$$m_i = \sum_p m_p S_{pi}(x_p^n) \tag{7}$$

It is important to note that the coefficients  $D_{pi}(x_p^n)$  and  $S_{pi}(x_p^n)$  (which here will be abbreviated to  $D_{pi}^n$  and  $S_{pi}^n$ , depend explicitly on the background mesh and the particle positions and that they also be chosen to reproduce derivatives of constant and linear functions exactly [5]. The initial volume of the particles is uniform for the  $n_p$  particles

in an interval. The particle volumes are defined using the deformation gradient,  $F_p^n$ , and the initial particle volume,  $V_p^0$ ,

$$V_p^n = F_p^n V_p^0, \text{ where } V_p^0 = \frac{h}{n_p}, \text{ where } F_p^0 = 1 \quad (8)$$

From (7) the continuous form of acceleration equation in the MPM method in this simple case is

$$a_i(t) = \frac{-1}{m_i} \sum_p D_{pi}(x_p(t)) \sigma_p(t) F_p(t) V_p^0 \quad (9)$$

The equation to update velocity at the nodes, as denoted by  $v_i^n$  is then given by

$$\dot{v}_i = a_i \quad (10)$$

The equation for the update of the particle velocities then

$$\dot{v}_p = a_p \quad (11)$$

where the value of the acceleration at a point  $x_p^n$  is given by interpolation based upon nodal values of acceleration

$$a_p = \sum_i S_{ip}(x_p(t)) a_i \quad (12)$$

The equation for the particle position updates is

$$\dot{x}_p = v_p \quad (13)$$

The update of the deformation gradients is given using,

$$\frac{\partial v}{\partial x}(x_p(t)) = \sum_i D_{ip}(x_p(t)) v_i \quad (14)$$

The deformation update equation is

$$\dot{F}_p = \frac{\partial v}{\partial x}(x_p(t), t) F_p \quad (15)$$

While the stress update equation is, using the appropriate constitutive model and Young's Modulus,  $E$ ,

$$\dot{\sigma}_p = E \frac{\partial v}{\partial x}(x_p(t)) \quad (16)$$

### 3 Stress Last MPM and Local Errors in Space and Time

In solving the system of equations defined above by equations (6) to (16) one standard approach used is to order the equations in a certain order and then to solve them in turn using explicit methods. Differences in how the equations are solved corresponds to whether or not the stress is updated first or last in a time step, a choice that is discussed at length by [1]. These two different choices are related to the use of the semi-implicit Euler A or B method, [7], [2]. Following Bardenhagen [1] it is preferable to increment

stress last. In this case it is assumed that at time  $t^n$  a consistent set of particle positions  $x_p^n$ , velocities  $v_p$ , stresses  $\sigma_p^n$  and deformation gradients  $F_p^n$  are available. The description here of the method and the sources of error follows that in [3]. It is also assumed that the GIMP method is use for spatial discretization The nodal velocity  $v_i$  is calculated using

$$v_i^n = \sum_p S_{pi}^n \frac{m_p}{m_i} v_p^n \quad (17)$$

with an associated local error  $v_{i,true}^n - v_i^n = EI_{pi}^n$  as defined below. The nodal acceleration is updated by using the stresses and deformation gradients at the current grid points

$$a_i^n = \frac{-1}{m_i} \sum_p D S_{pi}^n \sigma_p^n F_p^n V_p^0 \quad (18)$$

where the nodal mass is is defined by equation (7). The equation to update velocity at the nodes is then given by

$$v_i^{n+1} = v_i^n + dt a_i^n \quad (19)$$

The local error in this forward Euler step is given by

$$v_{i,true}^{n+1} - v_i^{n+1} = lev_i^{n+1} = \frac{dt^2}{2} \frac{d^2 v_i^n}{dt^2} + dt E a_i^n \quad (20)$$

where  $E a_i^n$  is the spatial error from using the approximation in equation (18), see Section 4 below and the time derivative term is the local error [7]. The value of the acceleration at a point  $x_p^n$  is given by interpolation based upon nodal values of acceleration:

$$a_p^n = \sum_i S_{ip}^n a_i^n \quad (21)$$

While there is no new temporal error here the error in acceleration  $E a_i^n$  is interpolated too and an additional interpolation error associated with the coefficients  $S_{ip}^n$  used in equation (21) is introduced, as denoted by  $E I_{ip}^n$ , see Section 4 below. The equation for the update of the particle velocity is then:

$$v_p^{n+1} = v_p^n + dt a_p^n \quad (22)$$

The associated local error is

$$v_{p,true}^{n+1} - v_p^{n+1} = lev_p^{n+1} = \frac{dt^2}{2} \frac{d^2 v_p}{dt^2} + dt E a_p^n + dt \sum_i S_{ip}^n E a_i^n \quad (23)$$

The velocity gradients at particles are calculated using the formula

$$\frac{\partial v^{n+1}}{\partial x}(x_p) = \sum_i D_{ip}^n v_i^{n+1} \quad (24)$$

with an associated differentiation error denoted by  $Ev_{xp}^{i+1}$ , see Section 4 below. These velocity gradients are used to update the stress and deformation gradients at particles

$$F_p^{n+1} = F_p^n + dt \frac{\partial v^{n+1}}{\partial x}(x_p^n, t_n) F_p^n dt \quad (25)$$

For the deformation gradient  $leF_p^{n+1}$  is the local time and space error given by

$$leF_p^{n+1} = \frac{dt^2}{2} \frac{d^2 F}{dt^2} + dt F_p^n \left( Ev_{xp}^n + \sum_i D_{ip}^n lev_i^{n+1} \right) \quad (26)$$

Stress is updated using the appropriate constitutive model and Young's Modulus,  $E$ ,

$$\sigma_p^{n+1} = \sigma_p^n + dt E \frac{\partial v^{n+1}}{\partial x}(x_p^n, t_n) \quad (27)$$

In this case the stress local time and space error,  $le\sigma_p^{n+1}$ , is given by

$$le\sigma_p^{n+1} = \frac{dt^2}{2} \frac{d^2 \sigma}{dt^2} + dt E \left( Ev_{xp}^n + \sum_i D_{ip}^n lev_i^{n+1} \right) \quad (28)$$

The equation for the particle position update is

$$x_p^{n+1} = x_p^n + dt v_p^{n+1} \quad (29)$$

For the particle update the local error as denoted by  $lex_p^{n+1}$  is given by

$$lex_p^{n+1} = \frac{dt^2}{2} \frac{d^2 x_p}{dt^2} + dt lev_p^{n+1} \quad (30)$$

This section shows that many of the material point errors result from the applications of the mapping matrix from particles to nodes  $S_{pi}$  (and its transpose  $S_{ip}$ ) and the differentiation matrix  $D_{pi}$  (and its transpose  $D_{ip}$ ).

#### 4 ESTIMATING THE SPATIAL ERROR

The above derivations illustrate how the spatial and temporal errors associated with MPM combine to give the overall error. Steffen et al. [12] observed these errors experimentally and arrived at the conclusion that for a stable time step with the methods they considered that temporal errors are dominated by spatial errors. The error framework presented in the previous section makes it possible to derive estimates for the individual parts of MPM associated with the mapping matrix  $S_{ip}$  and the differentiation matrix  $D_{ip}$  (and their transposes) and to thus provide computable estimates for the error. There are two parts to this process. The first part is that the error in mapping from particles to the grid. The second part is estimating the error in mapping from the grid back to particles.

#### 4.1 Particles to Nodes

Consider the mapping defined by equation (18). The first step is to expand the stress at a particle about the node by using a simple Taylor expansion.

$$\sigma_p = \sigma_i + (x_p - X_i) \frac{\partial \sigma}{\partial x}(X_i, t) + \frac{(x_p - X_i)^2}{2} \frac{\partial^2 \sigma}{\partial x^2}(X_i, t) + \frac{(x_p - X_i)^3}{6} \frac{\partial^3 \sigma}{\partial x^3}(X_i, t) + \dots \quad (31)$$

Substituting this in equation (18) gives after assuming that the coefficients  $D_{pi}$  exactly differentiate linear functions ( $\sum_p DS_{pi} = 0$  and  $\sum_p DS_{pi}x_p = 1$ ), see [5] who also provide a procedure for this.

$$Ea_i = \frac{-1}{\tilde{m}_i} \sum_p DS_{pi}^n \left[ \frac{(x_p - X_i)^2}{2} \frac{\partial^2 \sigma}{\partial x^2}(X_i, t) + \frac{(x_p - X_i)^3}{6} \frac{\partial^3 \sigma}{\partial x^3}(X_i, t) + \dots \right] \quad (32)$$

Truncating the series gives the estimate

$$Ea_i \approx \frac{-1}{\tilde{m}_i} \sum_p DS_{pi}^n \left[ \frac{(x_p - X_i)^2}{2} \frac{\partial^2 \sigma}{\partial x^2}(X_i, t) + \frac{(x_p - X_i)^3}{6} \frac{\partial^3 \sigma}{\partial x^3}(X_i, t) \right] \quad (33)$$

The estimation of these second and third derivatives at the nodes is described below. This is a first order error estimate as the coefficients  $D_{pi}$  differentiate the quadratic and cubic terms. However if the term  $\sum_p (x_p - X_i)$  is zero (e.g. symmetric particles about a node) then the error is second order.

In the case of the mapping defined by equation (17), (ignoring the contributions of the masses for the moment) using the approach of [5] it is assumed that the mapping is again linearity preserving i.e.  $\sum_p S_{pi} = 1$  and  $\sum_p S_{pi}x_p = X_i$ . A similar line of argument to that above gives

$$EI_{pi}^n \approx -\frac{\partial^2 v}{\partial x^2}(X_i, t) \sum_p S_{pi}^n \frac{(x_p - X_i)^2}{2} - \frac{\partial^3 v}{\partial x^3}(X_i, t) \sum_p S_{pi}^n \frac{(x_p - X_i)^3}{6} \quad (34)$$

where again cases we have to estimate higher order derivatives at the nodes.

#### 4.2 Nodes to Particles

In this case we have to consider two cases the first is the differentiation of velocity values at nodes to get velocity derivatives at particles given by equation (24), while the second is the mapping from nodal values of accelerations to accelerations at particles, given by equation (22). It is assumed that the transposes of the mapping matrices  $S$  and  $DS$  (as denoted by switching the subscript  $pi$  to  $ip$ ) satisfy the same equations as above for preserving linearity in the mapping and for differentiating linear functions exactly, e.g. using the procedure of Gritton [5]. In both these cases we expand the nodal values about particles. In the case of velocities we have from a Taylor expansion:

$$v_i^{n+1} = v_p^{n+1} + (X_i - x_p) \frac{\partial v}{\partial x}(x_p, t^{n+1}) + \frac{(X_i - x_p)^2}{2} \frac{\partial^2 v}{\partial x^2}(x_p, t^{n+1}) + \frac{(X_i - x_p)^3}{6} \frac{\partial^3 v}{\partial x^3}(x_p, t^{n+1}) + \dots \quad (35)$$

with a similar equation for acceleration. Substituting the right side of this in equation (21) and truncating the series gives

$$Ea_p^n = a_{p,true}^{n+1} - \sum_i S_{ip}^n a_i^{n+1} \quad (36)$$

where

$$Ea_p^n \approx - \sum_i S_{ip}^n \left( \frac{(X_i - x_p)^2}{2} \frac{\partial^2 a}{\partial x}(x_p, t^n) + \frac{(X_i - x_p)^3}{6} \frac{\partial^3 a}{\partial x^3}(x_p, t^n) \right) \quad (37)$$

This expression requires velocity derivatives at the particles. In the case of estimating the error in derivatives at particles a similar approach gives

$$Ev_{xp}^{n+1} = \frac{\partial v_{true}^{n+1}}{\partial x}(x_p) - \sum_i DS_{ip}^n v_i^{n+1}, \quad (38)$$

$$Ev_{xp}^{n+1} \approx - \sum_i DS_{ip}^n \left( \frac{(X_i - x_p)^2}{2} \frac{\partial^2 v}{\partial x^2}(x_p, t^{n+1}) + \frac{(X_i - x_p)^3}{6} \frac{\partial^3 v}{\partial x^3}(x_p, t^{n+1}) \right) \quad (39)$$

Again this expression requires velocity derivatives at the particles. These values may be approximated by interpolating from the nodal derivatives using say (21) so that, for instance,

$$\frac{\partial^2 v}{\partial x^2}(x_p, t) \approx \sum_i S_{ip} \frac{\partial^2 v}{\partial x^2}(X_i, t) \quad (40)$$

to get

$$Ev_{xp}^{n+1} \approx - \sum_i DS_{ip}^n \left( \frac{(X_i - x_p)^2}{2} \sum_i S_{ip} \frac{\partial^2 v}{\partial x^2}(X_i, t^{n+1}) + \frac{(X_i - x_p)^3}{3} \sum_i S_{ip} \frac{\partial^3 v}{\partial x^3}(X_i, t^{n+1}) \right) \quad (41)$$

### 4.3 Estimating the Spatial Derivatives

The second derivative of the stress is straightforwardly estimated using finite differences of nodal stress values.

$$\frac{\partial^2 \sigma}{\partial x^2}(X_i, t) \approx \frac{\sigma_{i+1} - 2\sigma_i + \sigma_{i-1}}{h^2} \quad (42)$$

and the third derivative similarly

$$\frac{\partial^3 \sigma}{\partial x^3}(X_i, t) \approx \frac{\sigma_{i+2} - 2\sigma_{i+1} + 2\sigma_{i-1} + \sigma_{i-2}}{h^3} \quad (43)$$

With appropriate modifications at the boundaries.

#### 4.4 Estimating the Local Stress Error

Consider the local stress error at the end of the step as given by equation (28) and ignore the temporal error contribution from that equation and from equation (20) to get.

$$le\sigma_p^{n+1} = dtE \left( Ev_{xp}^{n+1} + dt \sum_i D_{ip}^n Ea_i^n \right) \quad (44)$$

Of the two terms to consider,  $Ev_{xp}^{n+1}$  may be estimated by using equation (41) while the second term, involves the product of the matrices  $D_{ip}$  and  $D_{pi}$ . In order to simplify the testing of this we consider here the mapping from a velocity at particles to a spatial derivative at nodes to a second spatial derivative at particles. A key part of this is to ensure that this mapping satisfies derivative boundary conditions at the nodes. It is worth remarking that the formula involves replacing the velocity values by their spatial derivatives in equation (41), as we are estimating the error in the second derivative at the particles calculated by differentiating to the nodes and then differentiating back to the particles.

### 5 TESTING THE ESTIMATES OF THE SPATIAL ERROR

In order to test the estimates derived above a simple example velocity used in the vibrating bar example that is often the simplest MPM problem studied is used, e.g. [5]. The velocity is defined by

$$v(x, t) = C \sin(2\pi X) \cos(C\pi t), (x, t) \in [0, 1] \times (0, t_e) \quad (45)$$

where  $C = \sqrt{E}$  for a uniform density and  $E$  the Young's modulus (here  $E=1$ ) may vary greatly. The testing procedure used is to vary the number of nodes and the number of particles. Suppose that the error in the stress at a spatial point  $x$  is given by  $e\sigma(x)$ , then the following definitions are needed.

$$e\sigma_N = [e\sigma(X_1), \dots, e\sigma(X_N)]^T \quad (46)$$

$$e\sigma_p = [e\sigma(x_1), \dots, e\sigma(x_{np})]^T \quad (47)$$

with obvious extensions to errors in velocity  $ev_N$  and other quantities. Suppose that the estimated error is similarly denoted by  $e_{st}\sigma_p$ . The  $L_\infty$  vector norm is used. The error index of the estimated error norm is given by

$$EI_k = \frac{\|e_{st}\sigma_N\|_\infty}{\|e\sigma_N\|_\infty} \quad (48)$$

Where the index  $k$  refers to the particular error quantity that is being estimated. Case 1:  $EI_1$  the error index of the estimate given by equation (17) and so is the second-order error in mapping the solution from particles to nodes. Case 2:  $EI_2$  the error index of the estimate given by equation (18) with evenly spaced particles and so is the second-order error in mapping from the solution at particles to its derivative at nodes. Case 3:



**Table 1:** Error Norms and Error Indices for Model Problem Solution

Case	Points/ Interval	Error Norms: Intervals N=				Error Indices N=			
		21	41	81	161	21	41	81	161
1	4	6.5e-3	1.6e-3	4.1e-4	1.0e-4	0.99	0.99	1.0	1.0
	8	6.0e-3	1.5e-3	3.7e-4	9.4e-5	0.99	0.99	1.0	1.0
	8	5.8e-3	1.4e-3	3.7e-4	9.2e-5	0.99	0.99	1.0	1.0
	16	5.8e-3	1.4e-4	3.6e-4	9.1e-5	0.99	0.99	1.0	1.0
2	2	3.2e-2	8.0e-3	2.0e-3	5.0e-4	0.98	0.99	1.0	1.0
	4	3.5e-2	8.8e-3	2.2e-3	5.5e-4	0.98	0.99	1.0	1.0
	8	3.6e-2	9.1e-3	2.3e-4	5.6e-4	0.98	0.99	1.0	1.0
	16	3.6e-3	9.1e-3	2.3e-4	5.7e-4	0.98	0.99	1.0	1.0
3	2	6.5e-3	1.6e-3	4.1e-4	1.0e-4	0.98	5.2	5.3	5.3
	4	8.1e-3	2.0e-3	5.2e-4	1.3e-4	0.98	4.1	4.4	4.2
	8	8.5e-3	2.1e-3	5.4e-4	1.3e-4	0.98	0.99	1.0	1.0
	16	8.6e-3	2.1e-3	5.4e-4	1.3e-4	0.98	0.99	1.0	1.0
4	2	3.5e-1	1.7e-1	8.7e-2	4.4e-2	0.98	1.0	1.0	1.0
	4	5.2e-1	2.6e-1	1.3e-1	6.5e-2	0.99	1.0	1.0	1.0
	8	6.0e-1	3.0e-1	1.5e-1	7.6e-2	0.99	1.0	1.0	1.0
	16	6.5e-1	3.3e-1	1.6e-1	8.2e-2	0.99	1.0	1.0	1.0
5	2	2.2	4.6	2.5	1.3	0.75	3.0	3.9	3.3
	4	3.5	4.6	2.5	1.1	0.88	2.3	2.5	2.5
	8	4.8	2.3	1.1	5.7e-1	0.94	1.0	1.1	1.1
	16	4.6	2.5	1.2	6.4e-1	0.97	1.0	1.0	1.0

$EI_3$  the error index of the estimate given by equation (21) and so is the second-order error in mapping from the solution at nodes to particles. Case 4:  $EI_4$  the error index of the estimate given by equation (24) and so is the first-order error in mapping from the solution at nodes to its derivative at particles. Case 5: In this case  $EI_5$  is the error index of the combined error of applying the derivative matrix  $D_{pi}$  and its transpose  $D_{ip}$  to the particle velocity given by equation (45) to calculate the second derivative at the particles is estimated. In estimating these errors and the error indices, the analytical solution given above is used at  $t = \pi/4$ , The number of intervals used was 20,40,80 and 160 while in each case 1,2,4 and 8 evenly-spaced material points were used per interval. These results show that the error estimators we have developed initially appear to do a good job of estimating the errors as is shown by the error indices close to 1.

## 6 STABILITY CONTROL IN MPM

### 6.1 Stability Condition

The stability of the MPM is complicated as the method is inherently nonlinear and as the points move then the matrices used to map between particles and nodes and vice versa change with every time step potentially. Hence while linear stability theory may be used to provide some insight as in [8] a full analysis must be nonlinear [9, 2]. The problem

of variable coefficients is considered by [6], but the whole topic of nonlinear stability is a very complex issue. In terms of what is done computationally, the simplest approach used for the stability of the MPM is the speed of sound heuristic that gives rise to the equation:

$$E \frac{dt_{sound}^2}{h^2} \leq 1 \quad (49)$$

In the form of MPM given above the only place that  $E$  appears is in the equation for stress,  $\sigma$ . Suppose that we consider only stress errors and suppose that each step starts with an error  $\delta\sigma_p^n$  at each particle. The acceleration is updated with these two values which are the updated stresses and deformation gradients at the current grid points

$$\delta a_i^n = \frac{-1}{m_i} \sum_p DS_{pi}^n \delta\sigma_p^n F_p^n V_p^0 \quad (50)$$

where the nodal mass is defined by equation (7). The equation to update velocity at the nodes is then given by

$$\delta v_i^{n+1} = dt \delta a_i^n \quad (51)$$

The velocity gradient errors at particles are calculated using the formula

$$\frac{\partial \delta v^{n+1}}{\partial x}(x_p) = \sum_i DS_{ip}^n \delta v_i^{n+1} \quad (52)$$

Although these velocity gradients are used to update the stress and deformation gradients at particles here only the stress is considered. While stress is updated using the appropriate constitutive model and Young's Modulus,  $E$ ,

$$\delta\sigma_p^{n+1} = \delta\sigma_p^n + dt E \frac{\partial \delta v^{n+1}}{\partial x}(x_p^n, t_n) \quad (53)$$

Combining the above equations gives

$$\frac{\partial \delta v^{n+1}}{\partial x}(x_p^n, t_n) = \sum_i DS_{ip}^n dt \frac{-1}{m_i} \sum_p DS_{pi}^n \delta\sigma_p^n F_p^n V_p^0, \quad (54)$$

Substituting this in equation (53) gives

$$\delta\sigma_p^{n+1} = \delta\sigma_p^n - E dt^2 \sum_i DS_{ip}^n \frac{1}{m_i} \sum_p DS_{pi}^n \delta\sigma_p^n F_p^n V_p^0, \quad (55)$$

Which may be written in matrix form as

$$\delta\sigma_p^{n+1} = \left[ I - E \frac{dt^2}{h^2} D^* F \right] \delta\sigma_p^n \quad (56)$$

where  $F$  is a diagonal matrix with  $p$ th diagonal entry  $F_p^n V_p^0$  and  $D^* = (DS)^T M DS$  where  $DS$  is the matrix with entries  $h DS_{pi}^n$  and  $M$  is a diagonal matrix with  $i$ th diagonal entry

$1/m_i$ . The standard stability requirement is [9] for the quantity  $\left\| \left[ I - E \frac{dt^2}{h^2} D^* F \right]^k \right\|$  to be bounded as  $k \rightarrow \infty$ , assuming that this matrix is constant, which is only the case if the MPM particles do not move. This decomposition immediately does lead to a speed of sound type condition as a small speed of sound parameter  $E \frac{dt^2}{h^2}$  will limit the norm of this matrix.

## 6.2 Connection to Error Estimates

There is a connection from the stability condition above to the error estimates derived above in that in the expression for  $le\sigma_p^{n+1}$ , the term

$$E \frac{dt^2}{h^2} D^* \Delta\sigma = dt^2 E \sum_i D_{ip}^n E a_i^n \quad (57)$$

Where  $\Delta\sigma$  is simply the term in [...] in the right side of equation (33). Hence there is a simple estimate for the norm of the amplification matrix given by using the standard definition of a matrix norm that is subordinate to a vector norm by

$$\left\| \left[ I - E \frac{dt^2}{h^2} D^* F \right] \right\| \geq \frac{|\Delta\sigma - dt^2 E \sum_i D_{ip}^n E a_i^n|}{|\Delta\sigma|} \quad (58)$$

This expression shows how the spatial error estimate derived above may be viewed as being a lower bound for the norm of the stability matrix. This may make it possible to track stability automatically.

## 7 CONCLUSIONS

The result of the approach presented here is that a decomposition of the different errors in MPM has been used to derive estimates for the mappings inherent in MPM between particles and nodes and vice versa. These simple estimates have been shown to work well in the simple demonstration case used here. The connection to the standard speed of sound stability approach has also been demonstrated. Future work involves applying this approach in a full MPM simulation and for two and three space dimensions. As the estimates derived here involve approximating derivatives of solution values on a regular mesh, this would seem to be entirely possible.

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