Hierarchical Geodesic Models of Longitudinal Shape

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September 18, 2014
Longitudinal Shape Analysis

Goal: Understand how individuals change over time.

OASIS data:
- 72 healthy subjects
- 64 dementia subjects
- 2-5 images ~1 year apart

http://www.oasis-brains.org
**Linear Mixed-Effects Models**

Subject-level: \( y_i = X_i\beta + Z_i b_i + \epsilon \)

Group-level: \( b_i \sim N(0, \Lambda) \)

Data matrices: \( X_i, Z_i \), typically with \( Z_i \) a subset of \( X_i \)

**Fixed Effects** \( (\beta) \): coefficients shared by all individuals

**Random Effects** \( (b_i) \): perturbation of \( i \)th individual

Laird and Ware, *Biometrics*, 1982
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Estimation by EM algorithm (\( b_i \) are latent variables)

Laird and Ware, *Biometrics*, 1982
Scalar Data Example:

- Dependent variable: Right hippocampal volume
- Fixed effects: intercept, age slope, group effect
- Random effects: intercept

```latex
> lmeExample = lme(RightHippoVol ~ Age * Group,
+                    random = ~1 | ID, data = ldat)
```
OASIS Longitudinal Hippocampus Data

- RightHippoVol
- Age
  - 60 70 80 90
  - 2000 3000 4000 5000

Legend:
- Red: Nondemented
- Blue: Demented
Why Hierarchical Models?

Dependent variable, y

Independent variable, t
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Dependent variable, $y$

Independent variable, $t$
Why Hierarchical Models?
Shape Representations

Structure Boundaries
(Kendall’s Shape Space)

Image Deformations
(Diffeomorphisms)

\[ I(x) \rightarrow I \circ \phi^{-1}(x) \]
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(Kendall’s Shape Space)

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(Diffeomorphisms)

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In both cases, data live on a high-dimensional, nonlinear manifold.
Regression on Manifolds

Given:
Manifold data: $y_i \in M$
Scalar data: $x_i \in \mathbb{R}$

Want:
Relationship $f : \mathbb{R} \rightarrow M$
“how $x$ explains $y$”
Regression on Manifolds

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\[
\hat{f} = \arg \min_{f} \sum_{i=1}^{N} d(f(x_i), y_i)^2
\]

This is a least squares problem.
Geodesic Regression

- Generalization of linear regression.
- Least-squares fitting of geodesic to the data \((x_i, y_i)\).

\[
(\hat{p}, \hat{v}) = \arg \min_{(p, v) \in TM} \sum_{i=1}^{N} d(\text{Exp}(p, x_i v), y_i)^2
\]

Fletcher, MFCA 2011, IJCV 2013; Niethammer et al., MICCAI 2011
Hierarchical Geodesic Models for Longitudinal Data

- **Group Level**: Average geodesic trend \((\alpha, \beta)\)
- **Individual Level**: Trajectory for \(i\)th subject \((p_i, u_i)\)

Muralidharan et al., CVPR 2012; Singh et al., IPMI 2013
Comparing Geodesics: Sasaki Metrics

What is the distance between two geodesic trends?

Define distance between initial conditions:

\[ d_S((p_1, u_1), (p_2, u_2)) \]

Sasaki geodesic on tangent bundle of the sphere.
Hierarchical Model Using The Sasaki Metric

\[ y_{ij} = \text{Exp}(\text{Exp}(p_i, x_{ij} u_i), \epsilon_{ij}) \quad \text{Individual Level} \]

\[ (p_i, u_i) = \text{Exp}_S((\alpha, \beta), (v_i, w_i)) \quad \text{Group Level} \]

where \( \text{Exp} \) is the exponential map on \( M \) and \( \text{Exp}_S \) is the exponential map on the tangent bundle \( TM \), with respect to the Sasaki metric on \( TM \).
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- This is feasible for \textbf{finite-dimensional} manifolds.
- \textbf{Diffeomorphisms}, not so much.
Results on Longitudinal Corpus Callosum

Permutation Test:

<table>
<thead>
<tr>
<th>Variable</th>
<th>$T^2$</th>
<th>$p$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept $\alpha$</td>
<td>0.734</td>
<td>0.248</td>
</tr>
<tr>
<td>Slope $\beta$</td>
<td>0.887</td>
<td>0.027</td>
</tr>
</tbody>
</table>
HGM for Diffeomorphisms

- Individual level: $N$ geodesic regression problems
- Group level: One group geodesic, $I(0), m(0)$
Comparing Geodesics for Diffeomorphisms

Group level geodesic parameterization

- **Intercept:** Image: $I$
- **Slope:** Initial momenta field: $m = Lv$
### Comparing Geodesics for Diffeomorphisms

<table>
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<th>Group level geodesic parameterization</th>
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<tbody>
<tr>
<td><strong>Intercept:</strong> Image: $I$</td>
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<td><strong>Slope:</strong> Initial momenta field: $m = Lv$</td>
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<th>Transforming intercepts and slope</th>
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<tbody>
<tr>
<td>Group action on image: $\phi \cdot I = I \circ \phi^{-1}$</td>
</tr>
<tr>
<td>Group action on momenta: $\phi \cdot m(0) = \text{Ad}_{\phi^{-1}}^* m(0)$</td>
</tr>
</tbody>
</table>

Co-adjoint transport
Group Level Optimization Problem

\[ \mathcal{E}(m(0), I(0), p_i(0)) = \frac{1}{2} \left\| m(0) \right\|_K^2 + \frac{1}{2} \sum_{i=1}^{N} \left( \left\| p_i(0) \right\|_K^2 + \left\| \rho_i \cdot \psi(t_i) \cdot I(0) - J_i \right\|_{L^2} \right) + \frac{1}{2} \sum_{i=1}^{N} \left\| \rho_i \cdot \psi(t_i) \cdot m(0) - n_i \right\|_K^2. \]
Longitudinal Diffeomorphism Results