Polynomial Regression on Riemannian Manifolds

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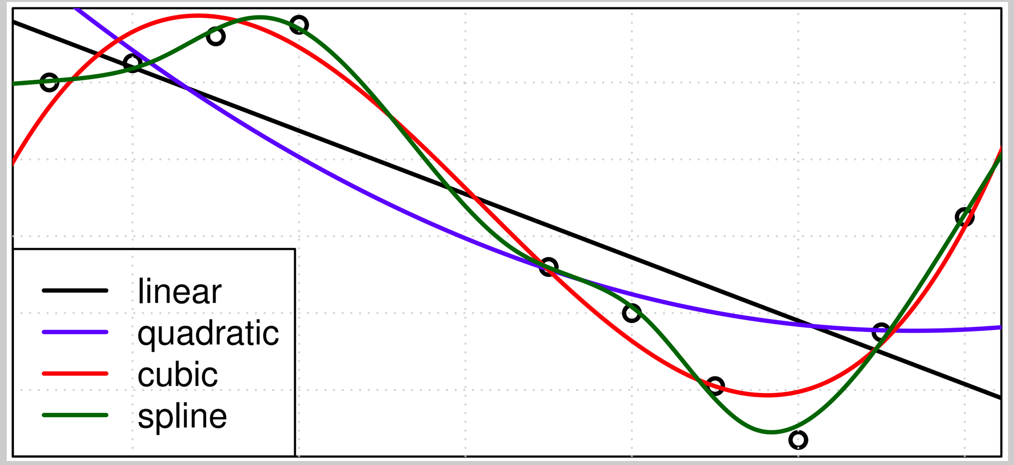
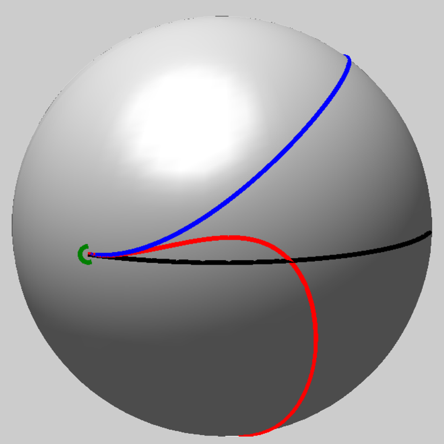
Parametric vs. Nonparametric Regression

- Nonparametric analysis includes kernel and spline-based curve fitting
- These result in a good fit, but number of parameters tied to amount of data
- Parametric regression (e.g. polynomial fitting) uses compact representation, resulting in more powerful statistical inference



Polynomials balance flexibility with simplicity

Riemannian Polynomials



Polynomials on the sphere

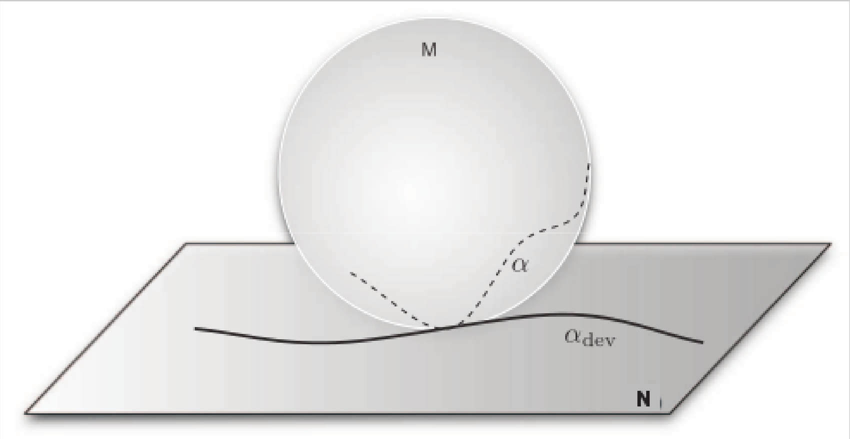


Figure: Leite & Krakowski 2008

- There are multiple ways to generalize polynomials to Riemannian manifolds
- Variational and energy-based generalizations are better suited to nonparametric regression schemes
- We use the rolling maps of Jupp & Kent to generalize polynomials
- The resulting constraint is the Riemannian polynomial equation:

$$\nabla^k_{\dot\gamma}\dot\gamma = 0$$

- "Rolling" the manifold against a flat Euclidean space without slipping or twisting, *Riemannian polynomials* trace out *Euclidean polynomials*

Riemannian polynomials provide a heirarchy of flexible intrinsic models for curve-fitting

References

- F. Bookstein. Shape and the Information in Medical Images. 1997.
- P. T. Fletcher. Geodesic Regression on Riemannian Manifolds. 2011.
- P. E. Jupp and J. T. Kent. Fitting smooth paths to spherical data. 1987.
- F. S. Leite and K. A. Krakowski. Covariant differentiation under rolling maps. 2008.

Adjoint Optimization on Manifolds

We minimize sum of squared geodesic distances

$$E = \sum_{j=1}^{N} d(\gamma(t_j), y_j)^2$$

Adjoint variables λ_i are initialized to zero at $t = 1$ and integrated backwards to $t = 0$:

$$\nabla_{\dot\gamma}\lambda_0 = -\sum_{i=1}^{k} R(v_i, \lambda_i)v_1 - \sum_{j=1}^{N}\delta(t - t_j)\,\mathrm{Log}_\gamma\, y_j$$

$$\nabla_{\dot\gamma}\lambda_1 = -\lambda_0 \qquad \dots \qquad \nabla_{\dot\gamma}\lambda_k = -\lambda_{k-1}$$

The adjoint variables are used to compute gradients with respect to initial conditions:

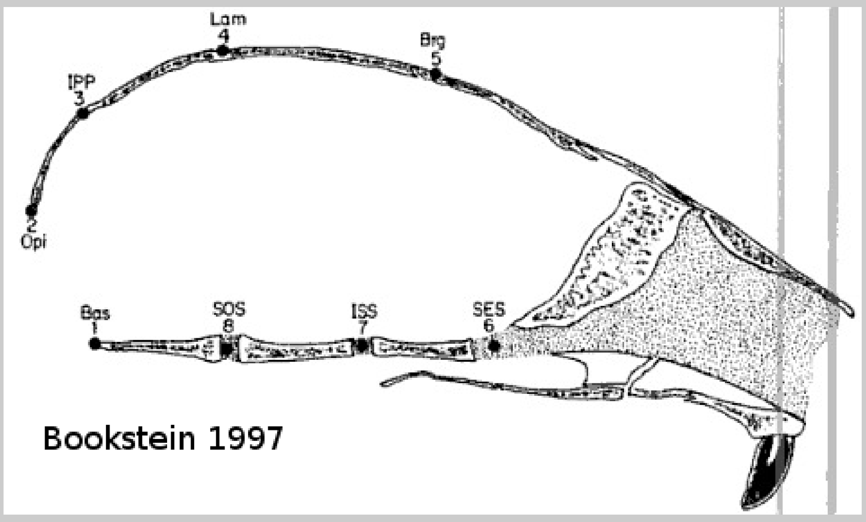
$$\delta_{\gamma(0)}E = -\lambda_0(0)$$
$$\delta_{v_1(0)}E = -\lambda_1(0)$$
$$\vdots$$
$$\delta_{v_k(0)}E = -\lambda_k(0)$$

These gradients are then used in a gradient descent scheme to minimize E.

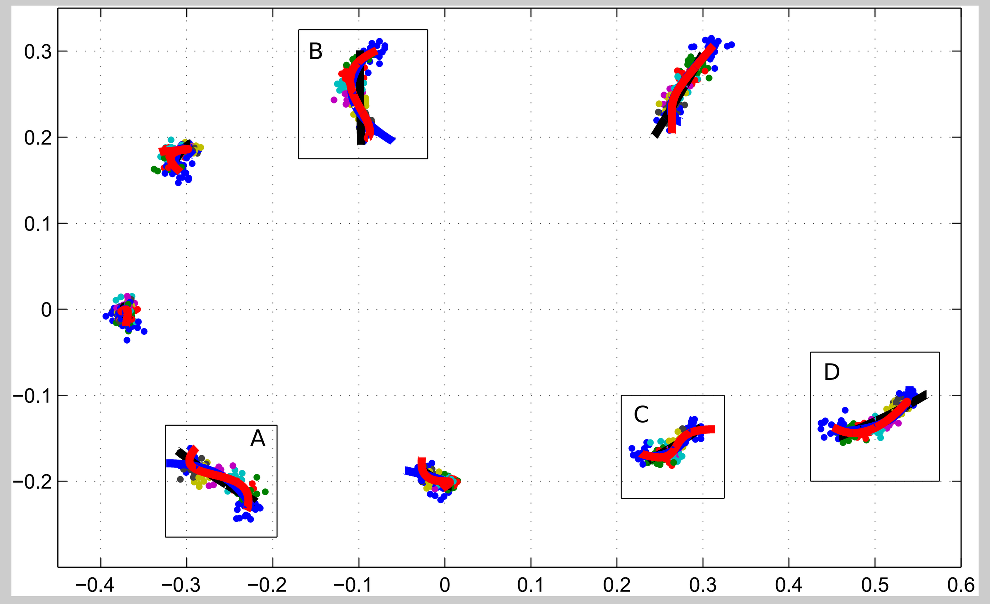
Kendall Shape Space

- Given sets of observed landmark positions, one often wants to study shape trends without the influence of position, scale, and rotation
- Kendall's shape space models this geometrically as a Riemannian manifold
- Shapes are represented as equivalence classes of point sets under similarity transformations
- For 2D shapes, Kendall shape space is isomorphic to complex projective space $\mathbb{C}P^{n-2}$

Results: Bookstein Rat Calivaria Growth

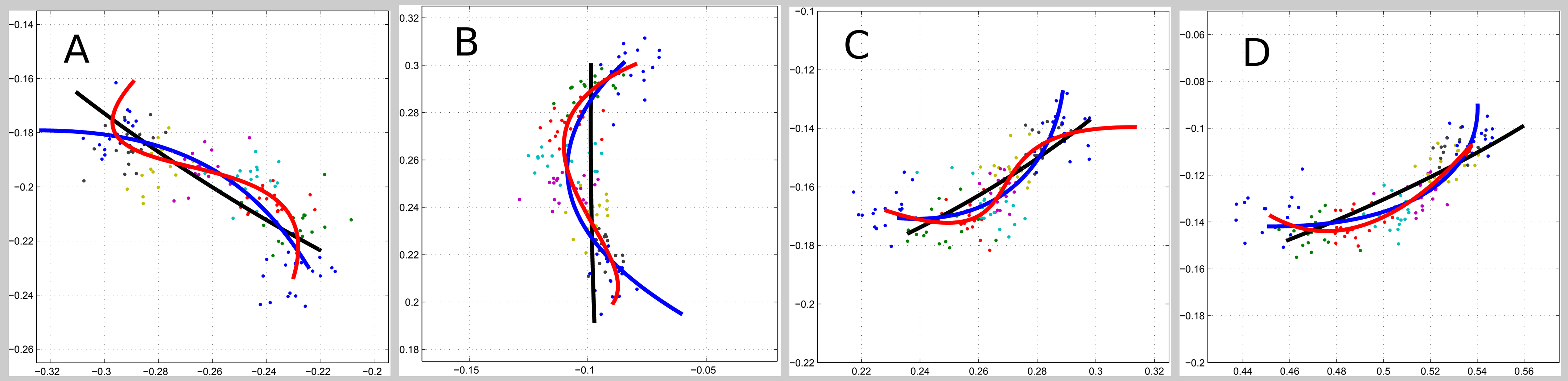


- 18 subjects, 8 landmarks, 8ages
- Polynomials fit in Kendall shape space



Polynomials of orders one (black, $R^2 = 0.79$), two (blue, $R^2 = 0.85$), and three (red, $R^2 = 0.87$).

Quadratic and cubic curves fit this data much better than geodesics

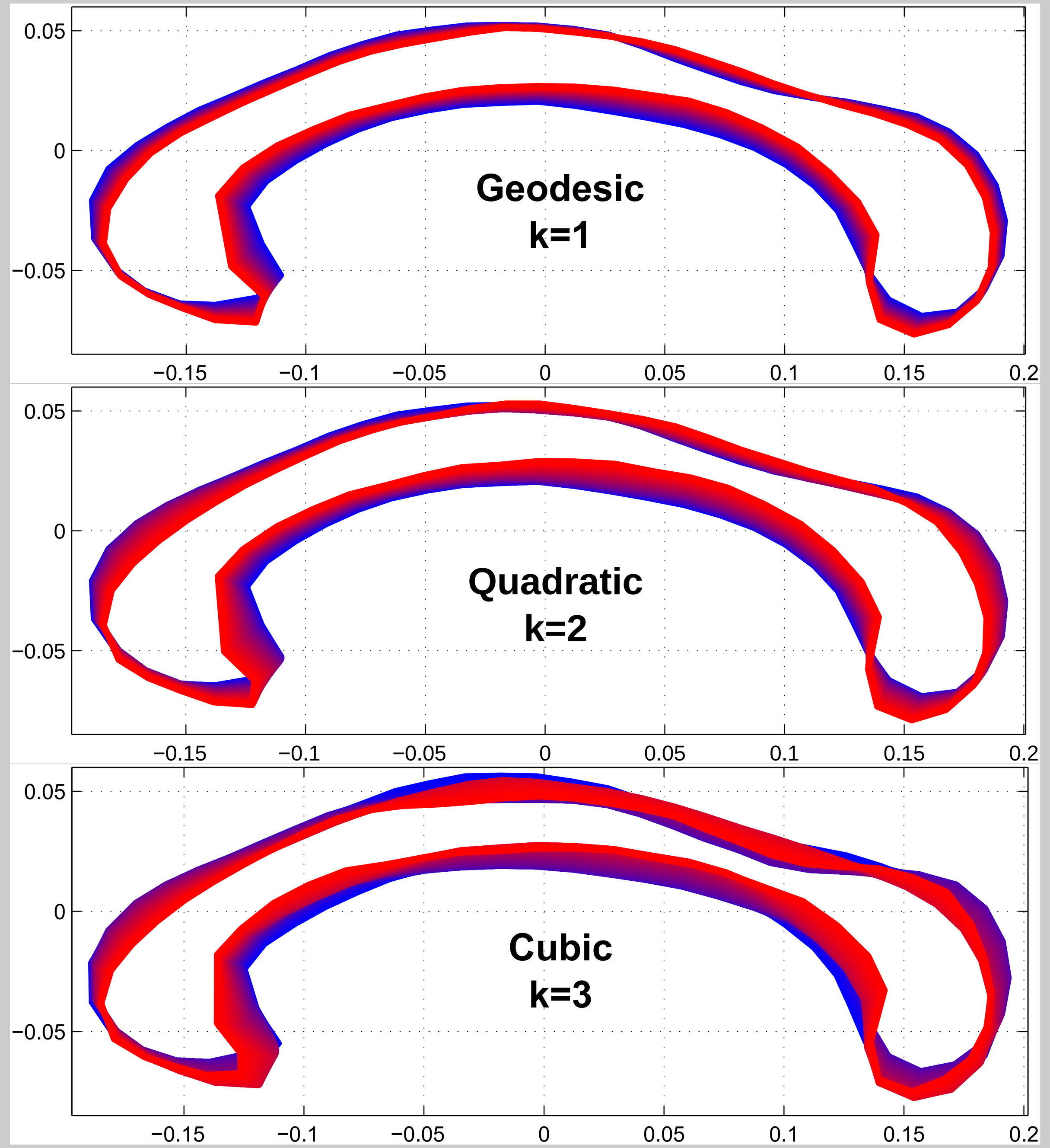


Zoomed views of individual landmark trajectories

Results: Corpus Callosum Aging

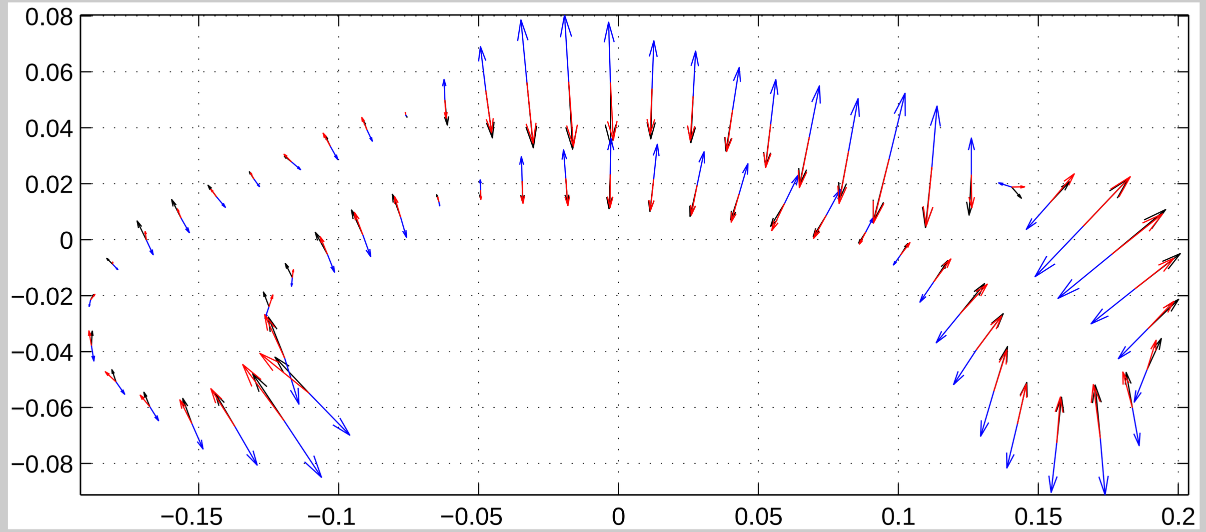
Figure: Fletcher 2011

- 32 subjects age 19–90 from OASIS brain database (`www.oasis-brains.org`)
- 64 landmarks, generated using ShapeWorks (`www.sci.utah.edu/software.html`)



Shape curves for geodesic (top, $R^2 = 0.12$), quadratic (middle, $R^2 = 0.13$), and cubic (bottom, $R^2 = 0.21$) regressions.

Collinear initial conditions imply time reparametrization of a geodesic path



Estimated initial conditions (black=velocity, blue=acceleration, red=jerk) for cubic regression.

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