Approximate Level-Crossing Probabilities for Interactive Visualization of Uncertain Isocontours

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Working with Uncertainty Workshop, VisWeek 2011

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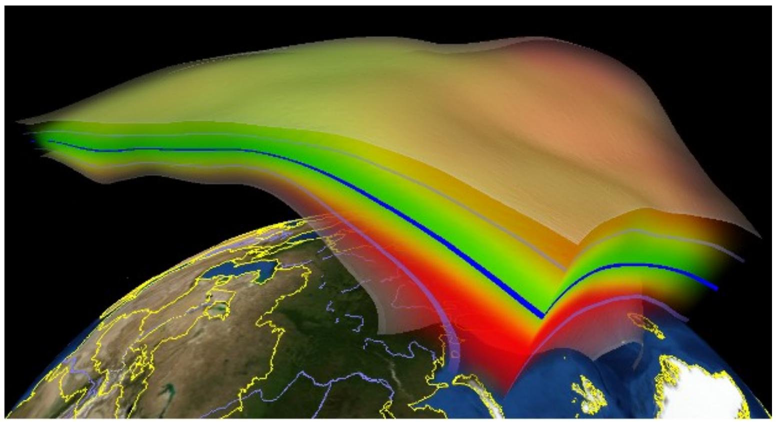
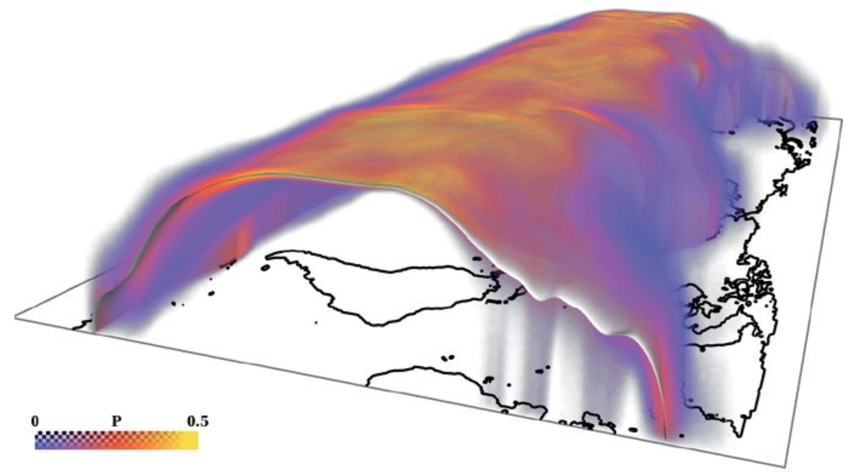
Deutsche Forschungsgemeinschaft

Previous Work

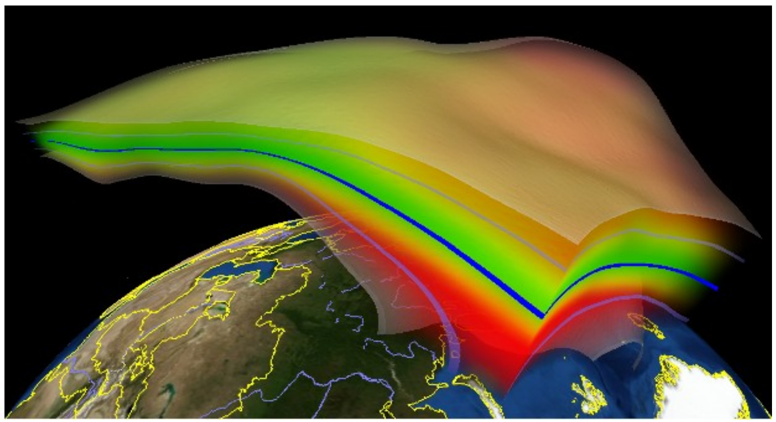


Pfaffelmoser, Reitinger & Westermann:
Visualizing the Positional and Geometrical Variability of Isosurfaces in Uncertain Scalar Fields,
EuroVis 2011



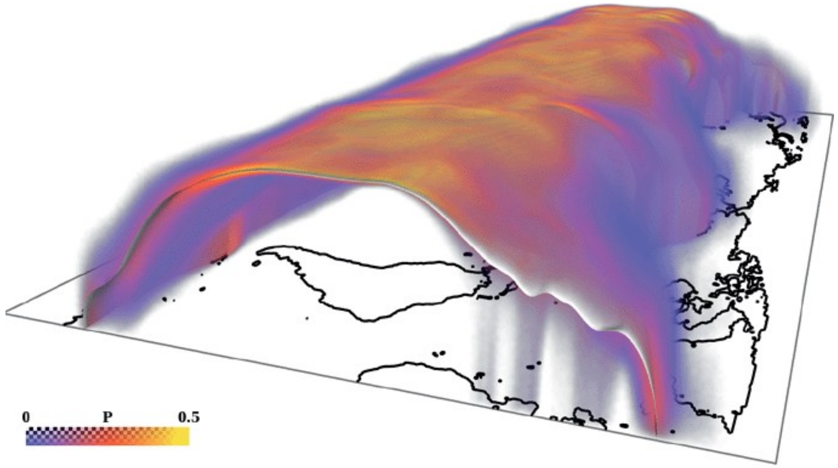
Pöthkow, Weber & Hege:
Probabilistic Marching Cubes,
EuroVis 2011

Previous Work



Pfaffelmoser, Reitinger & Westermann (2011)

- • First-crossing probabilities along rays
- ✔ Fast computation using lookup tables
- ✗ Fields with exponential correlation functions
- ✗ Depends on viewing-direction



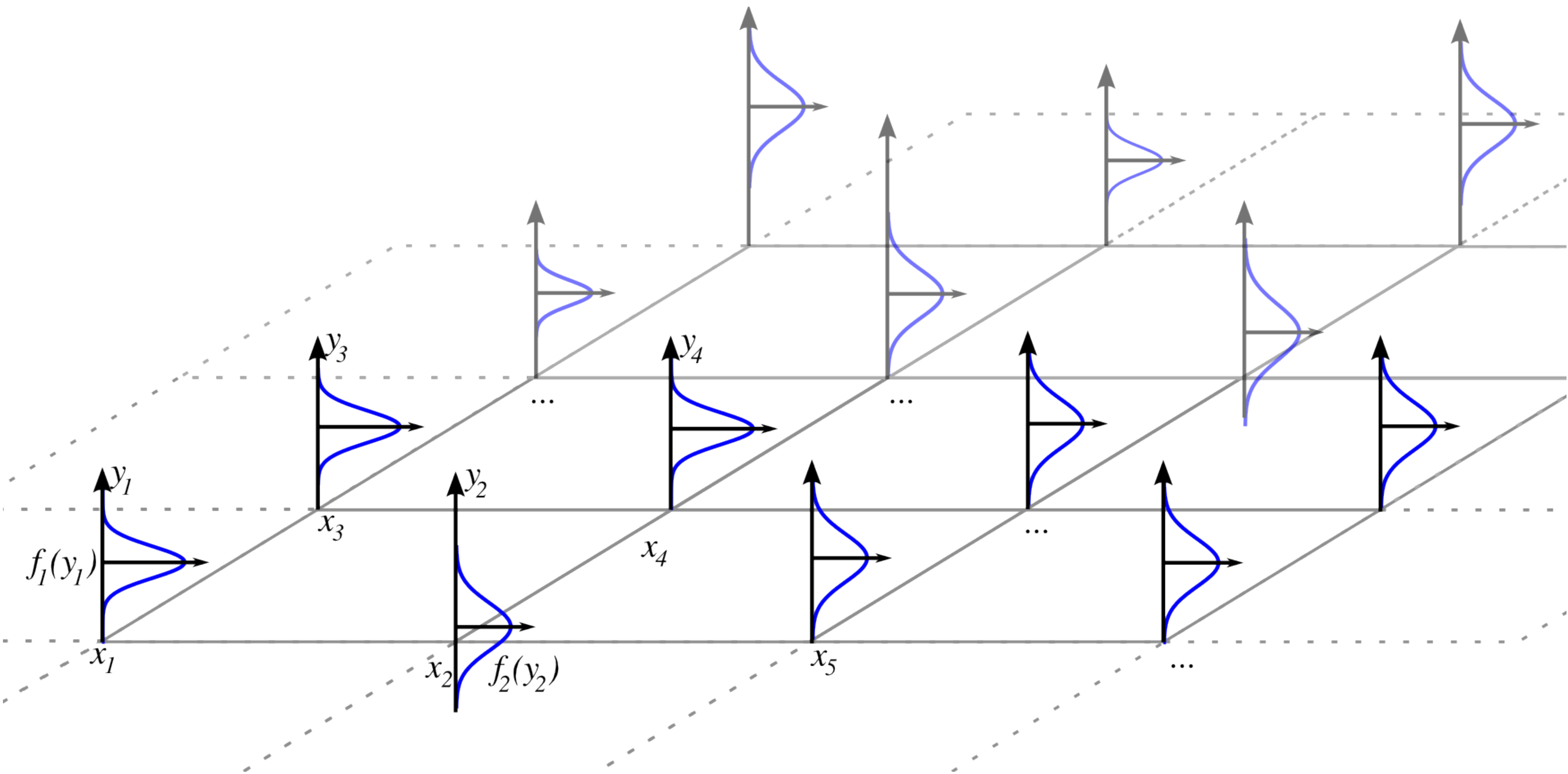
Pöthkow, Weber & Hege (2011)

- • Cell-wise level-crossing probabilities
- ✔ Arbitrary correlation structures
- ✗ Computationally expensive Monte-Carlo integration

Aims

- Fast and accurate approximation of cell-wise level-crossing probabilities
- Quantification of approximation errors
- Application to climate simulation data

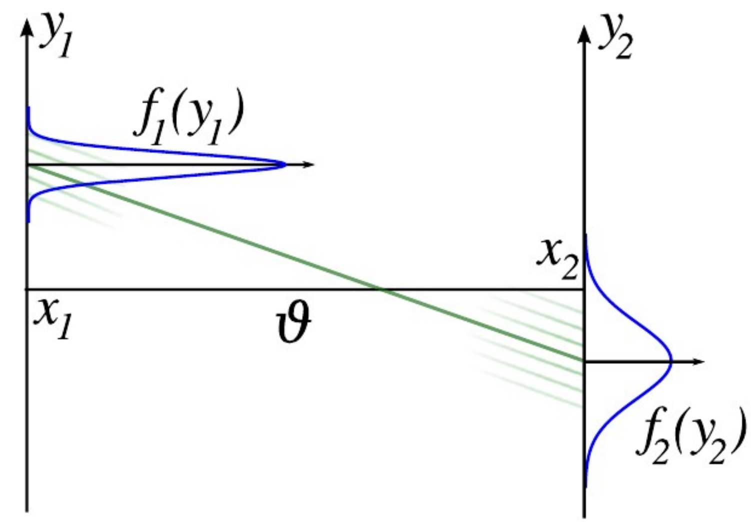
Input Data



$$\mu_i = \mathrm{E}(Y_i) \qquad \Sigma = [\mathrm{Cov}(Y_i, Y_j)]_{i=1,2,\ldots,n;\, j=1,2,\ldots,n}$$

Level Crossing Probabilities

Level Crossing Probabilities



- Assume 'local' interpolation & extremal values at grid points (e.g., linear, bi- or trilinear)
- Define $Y_i^+ = \{Y_i > \vartheta\}$
  $Y_i^- = \{Y_i \leq \vartheta\}$
- Level crossing probabilities (1D)

$$\mathrm{P}_c(\vartheta\text{-crossing}) = \mathrm{P}(Y_1^- \cap Y_2^+) + \mathrm{P}(Y_1^+ \cap Y_2^-)$$

$$= \int_{Y_1^-} \mathrm{d}y_1 \int_{Y_2^+} \mathrm{d}y_2\, f_{\mathbf{Y}}(y_1, y_2) + \int_{Y_1^+} \mathrm{d}y_1 \int_{Y_2^-} \mathrm{d}y_2\, f_{\mathbf{Y}}(y_1, y_2)$$

Level Crossing Probabilities

- Alternatively, use the complement

$$\mathrm{P}_c(\vartheta\text{-crossing}) = 1 - \mathrm{P}_c(\vartheta\text{-non-crossing})$$

- Non-crossing case: faster for >2 vertices

Standardization for Lookup-Table

- Parameter Transformation

$$F_{\mathbf{Y}}(y_1, y_2, \rho; \mu_1, \mu_2, \sigma_1, \sigma_2)$$

$$\Downarrow \quad \psi_i = \frac{\mu_i - \vartheta}{\sigma_i}$$

$$\tilde{F}_{\Psi}(\psi_1, \psi_2, \rho; 0, 0, 1, 1)$$

- Reformulated level-crossing probability

$$\begin{aligned} \mathrm{P}_c(\vartheta\text{-crossing}) &= 1 - \left(\mathrm{P}(Y_1^- \cap Y_2^-) + \mathrm{P}(Y_1^+ \cap Y_2^+)\right) \\ &= 1 - \left(\tilde{F}_{\Psi}(-\psi_1, -\psi_2, \rho) + \tilde{F}_{\Psi}(\psi_1, \psi_2, \rho)\right) \end{aligned}$$

→ 3D lookup-table for $\tilde{F}_{\Psi}(\psi_1, \psi_2, \rho)$

Level Crossing Probabilities

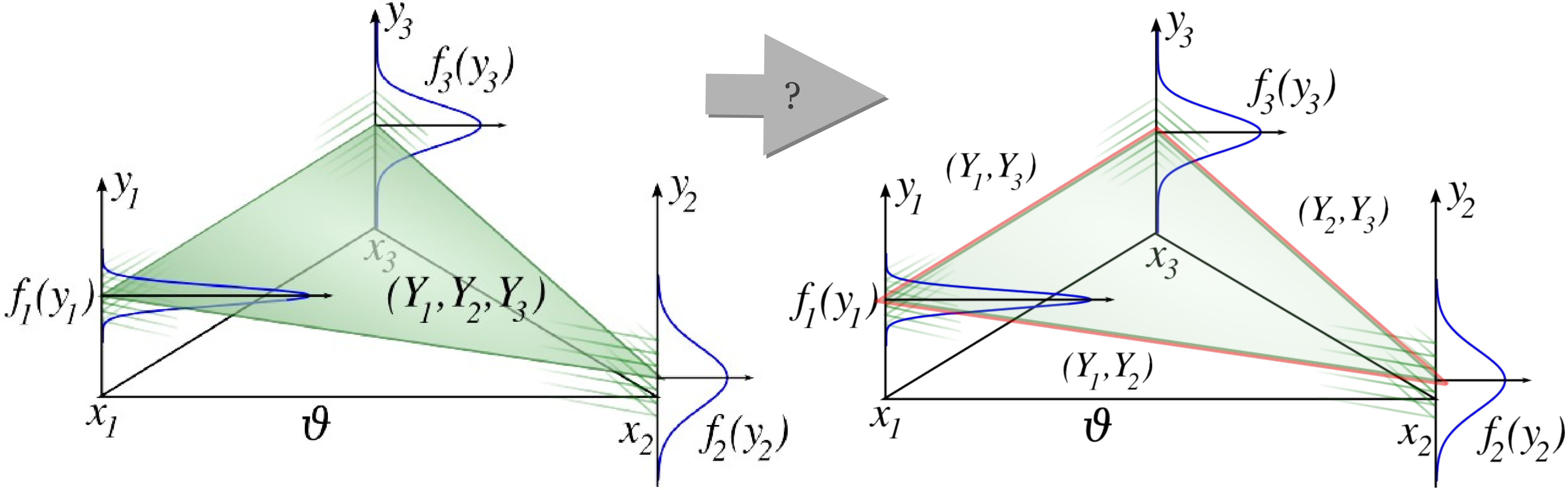
- n-dimensional case

$$\mathrm{P}_c(\vartheta\text{-crossing}) = 1 - \big(\mathrm{P}(Y_1^- \cap Y_2^- \ldots \cap Y_n^-) + \mathrm{P}(Y_1^+ \cap Y_2^+ \ldots \cap Y_n^+)\big)$$

- High-dimensional integration necessary

$$\mathrm{P}(Y_1^+ \cap Y_2^+ \ldots \cap Y_n^+) = \int_{Y_1^+} \mathrm{d}y_1 \int_{Y_2^+} \mathrm{d}y_2 \ \ldots \int_{Y_n^+} \mathrm{d}y_n \ f_{\mathbf{Y}}(y_1, y_2, \ldots, y_n)$$

Example: Triangular Cell



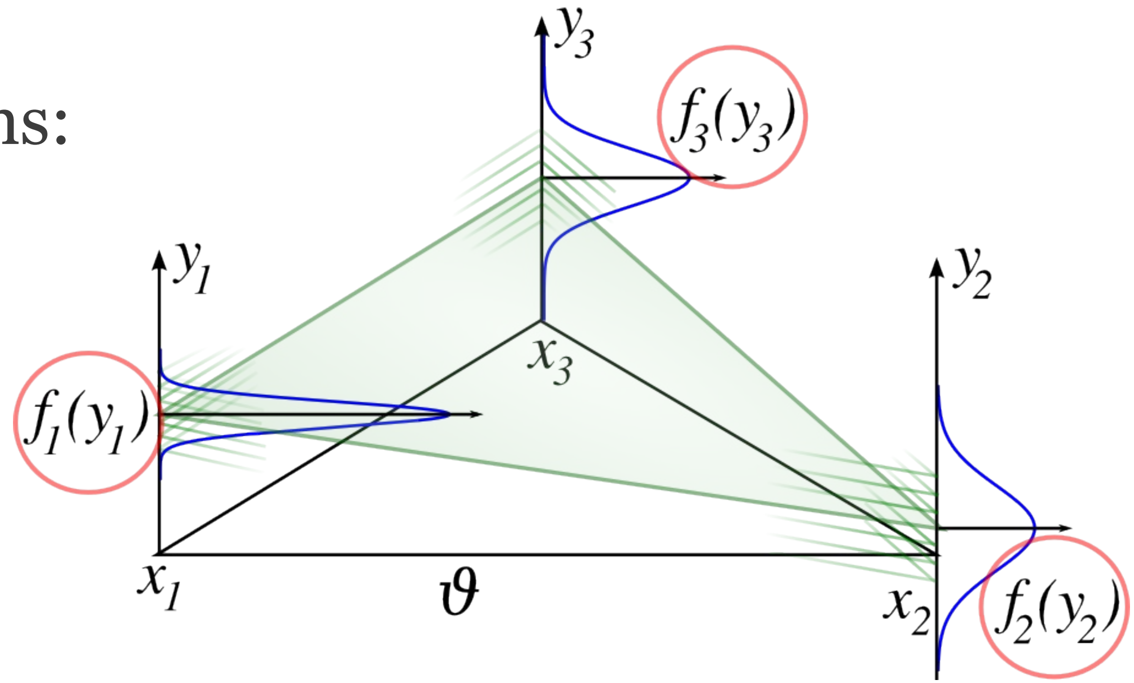
Approximation

Statistically Independent Vertices

- Approximate probability (cell c with n vertices): neglect correlation

$$Q_c = 1 - \left(P(Y_1^+)P(Y_2^+)\ldots P(Y_n^+) + P(Y_1^-)P(Y_2^-)\ldots P(Y_n^-)\right)$$

- For positive correlations: overestimates crossing probability
- Fast evaluation using 1D lookup-table

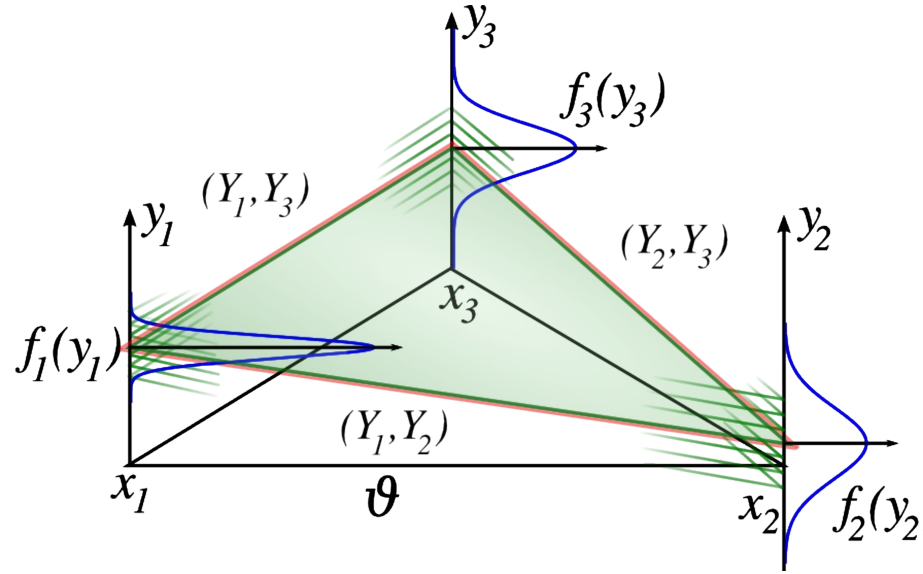


Maximum Edge Crossing Probability

- Cell-crossing ⟷ at least 2 edge-crossings
- Edge-crossing probabilities are lower bound for cell level-crossing probability
- Approximate probability (for m edges of a cell c)

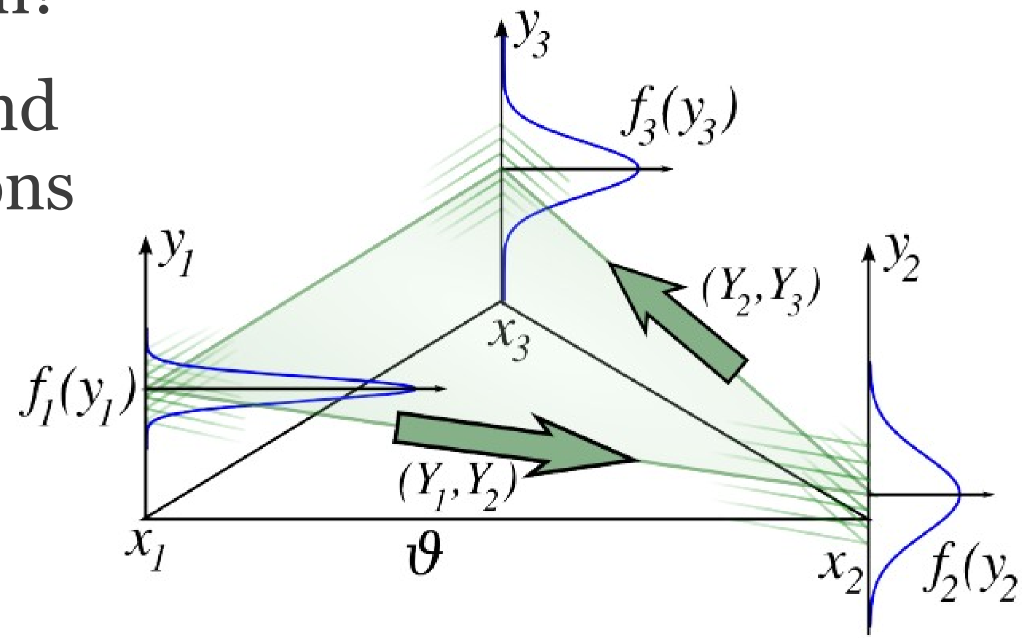
$$\mathrm{R}_c = \max_{i=1\ldots m} \left(1 - \left(\mathrm{P}(Y_{i,1}^{+} \cap Y_{i,2}^{+}) + (\mathrm{P}(Y_{i,1}^{-} \cap Y_{i,2}^{-})\right)\right)$$

- Fast evaluation using 3D lookup-table



Can we do better?

- Max. edge probability only considers 2D marginal distributions
- Can we utilize the other parameters of the *n*-dimensional distribution?
- Idea: traverse the cell and propagate the correlations

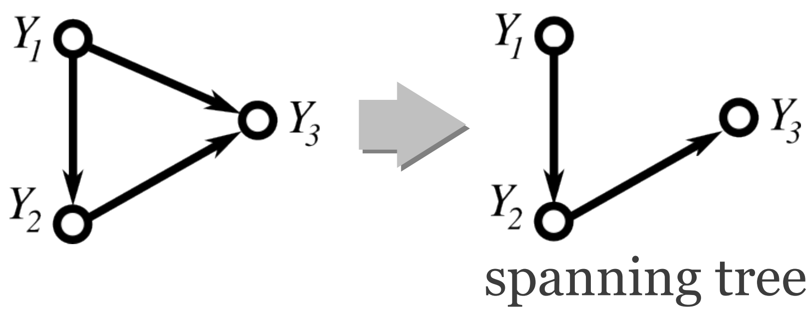


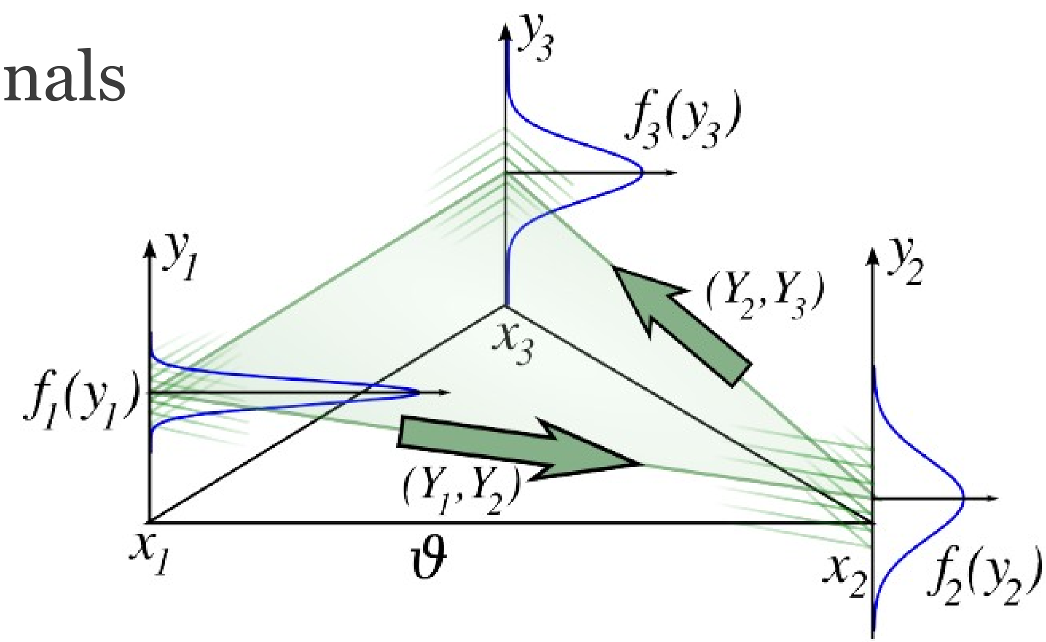
Linked-Pairs Approximation

- Approximate $\mathrm{P}(Y_1^+ \cap Y_2^+ \ldots \cap Y_n^+)$ using

$$\tilde{\mathrm{P}}(Y_1^+, Y_2^+, \ldots, Y_n^+) := \mathrm{P}(Y_1^+)\mathrm{P}(Y_2^+|Y_1^+)\,\mathrm{P}(Y_3^+|Y_2^+)\,\ldots\,\mathrm{P}(Y_n^+|Y_{n-1}^+)$$

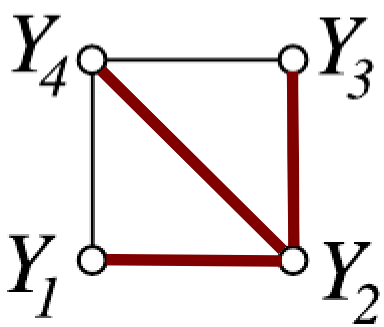
- No higher order conditionals are considered
- Graphical model

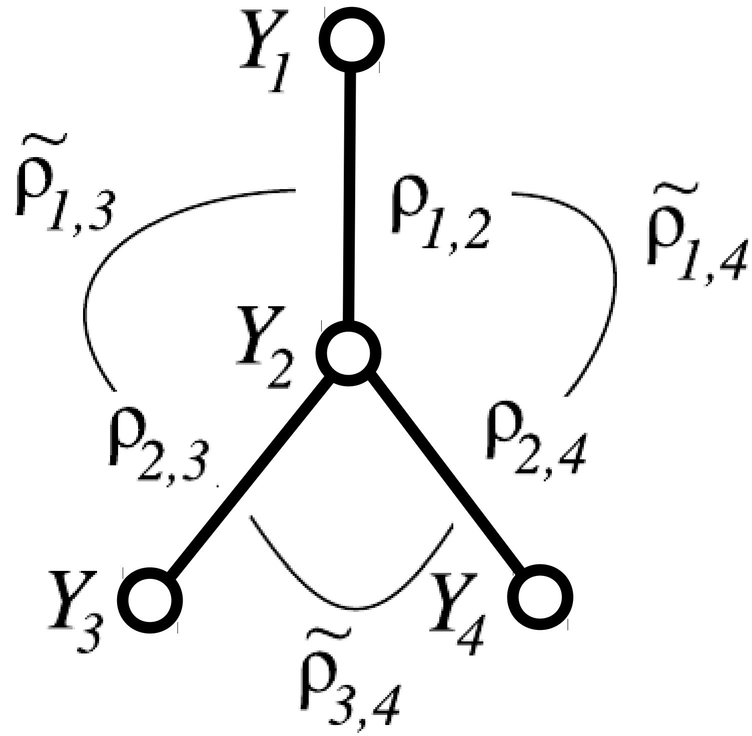




- Evaluation using tables (1D and 2D CDFs)

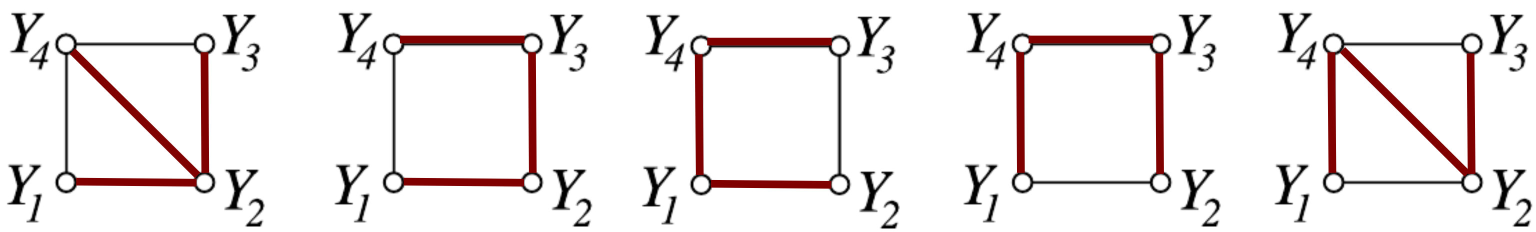
Linked-Pairs Approximation





$$\tilde{\mathrm{P}}(Y_1^+, Y_2^+, Y_3^+, Y_4^+) \;=\; \mathrm{P}(Y_1^+)\, \mathrm{P}(Y_2^+ | Y_1^+)\, \mathrm{P}(Y_3^+ | Y_2^+)\, \mathrm{P}(Y_4^+ | Y_2^+)$$

Spanning Trees

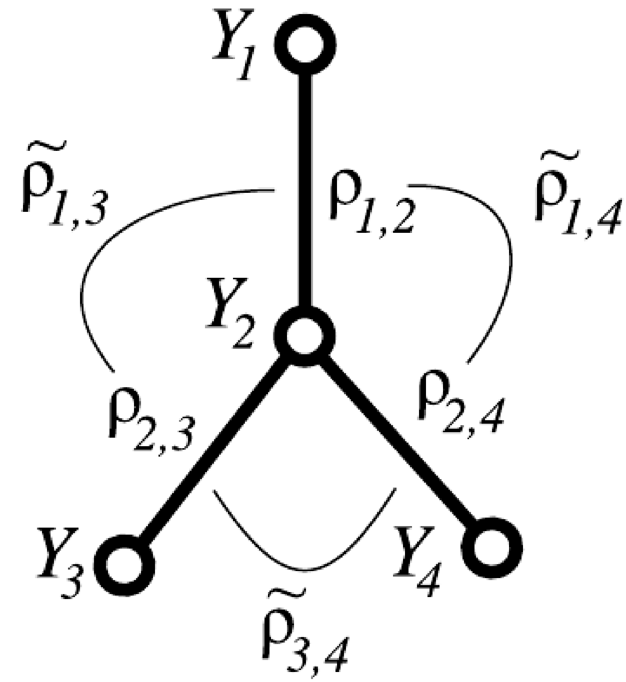


...

n^{n-2} trees (Cayley's formula)

Linked-Pairs Approximation

- Induced approximate distribution $\tilde{\mathbf{Y}} \sim \mathcal{N}(\mu, \tilde{\Sigma})$
  - Original expected values
  - Induced covariance matrix $\tilde{\Sigma}$ with $\tilde{\rho}_{\zeta,j} = \tilde{\rho}_{\zeta,i}\rho_{i,j}$



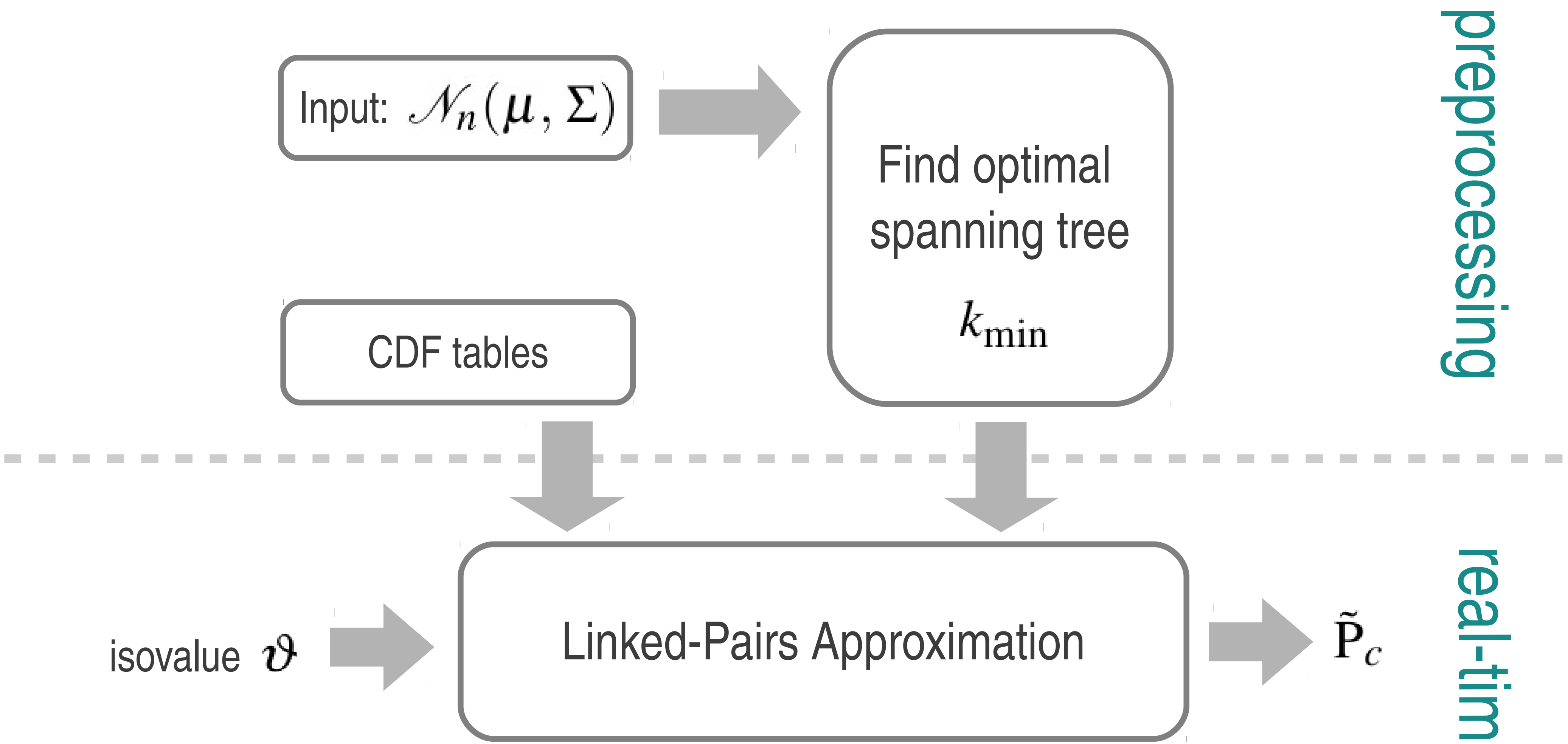
Optimal Approximate Distribution

- Distribution $\tilde{\mathbf{Y}} \sim \mathcal{N}(\mu, \tilde{\Sigma})$ depends on the spanning tree k
- Bhattacharyya distance to original distribution

$$D_B(k) = \frac{1}{2} \ln \left(\frac{\det((\Sigma + \tilde{\Sigma}_k)/2)}{\sqrt{\det(\Sigma)\,\det(\tilde{\Sigma}_k)}} \right)$$

- Choose

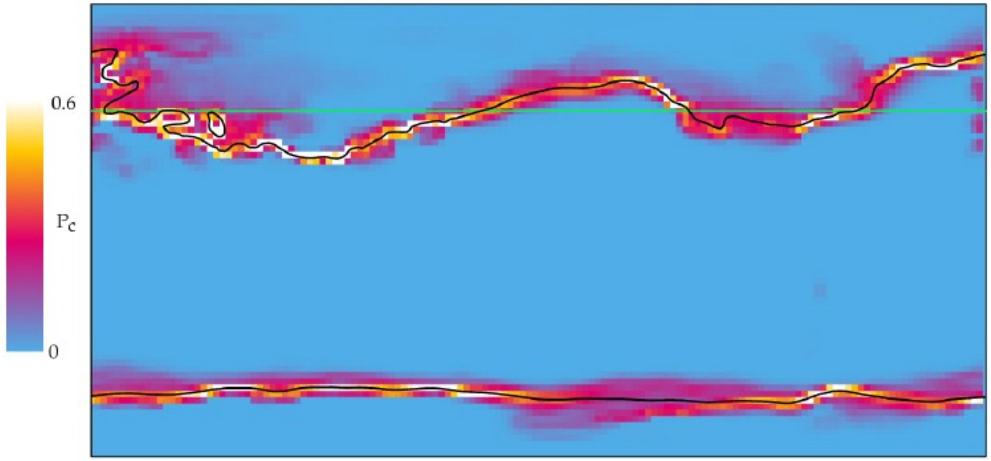
$$k_{\min} = \arg\min_k D_B(k)$$



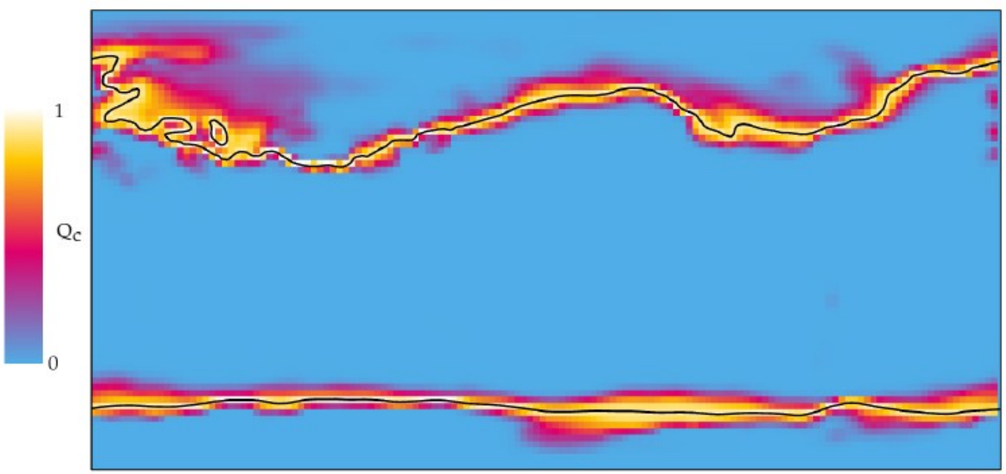
Input: $\mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$
Find optimal spanning tree
$k_{\min}$
CDF tables
preprocessing
isovalue ϑ
Linked-Pairs Approximation
$\tilde{\mathrm{P}}_c$
real-time

Results

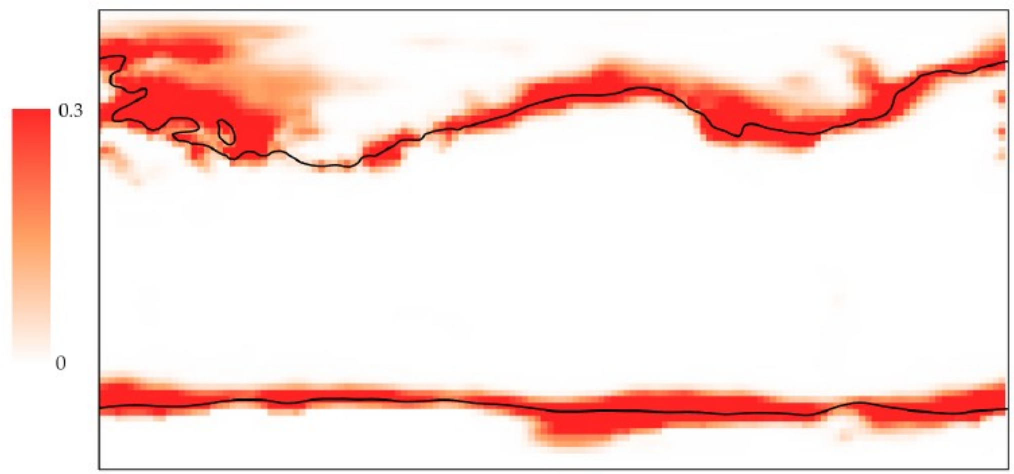
Climate Simulation Results



Monte Carlo integration (1000 samples)

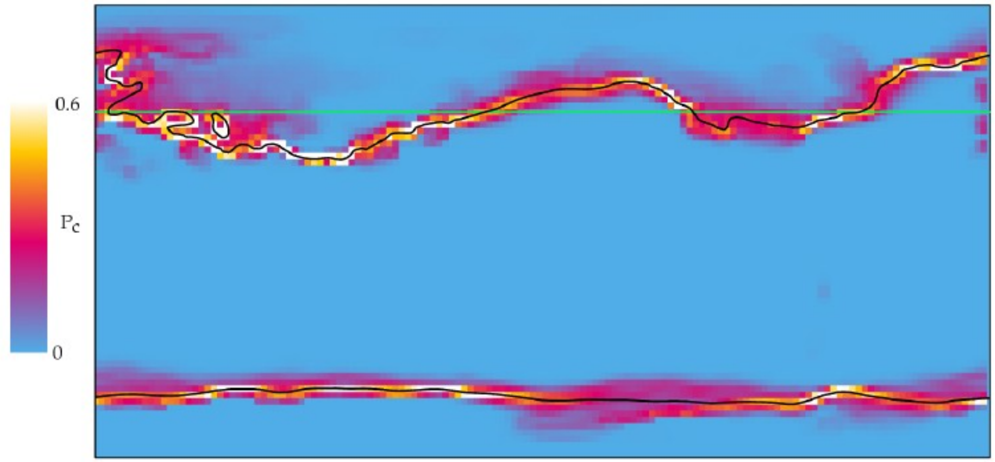


independent vertices (no correlation consideres)

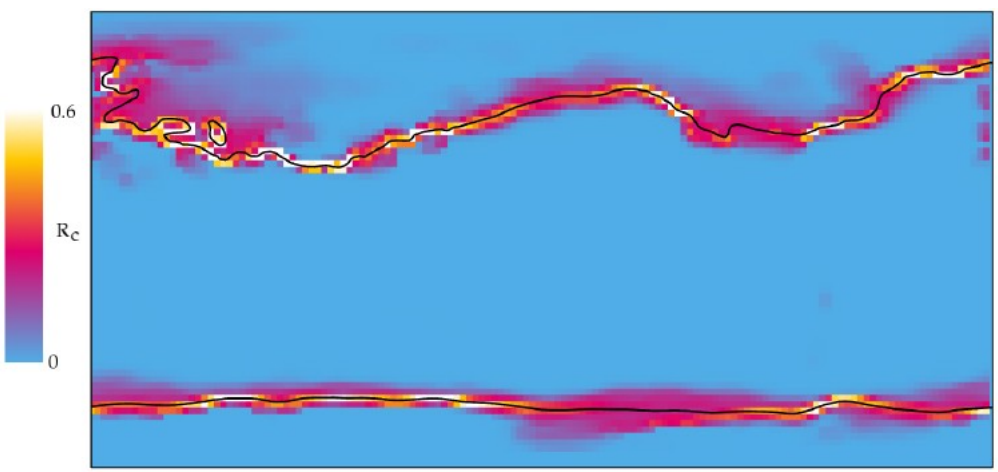


absolute differences

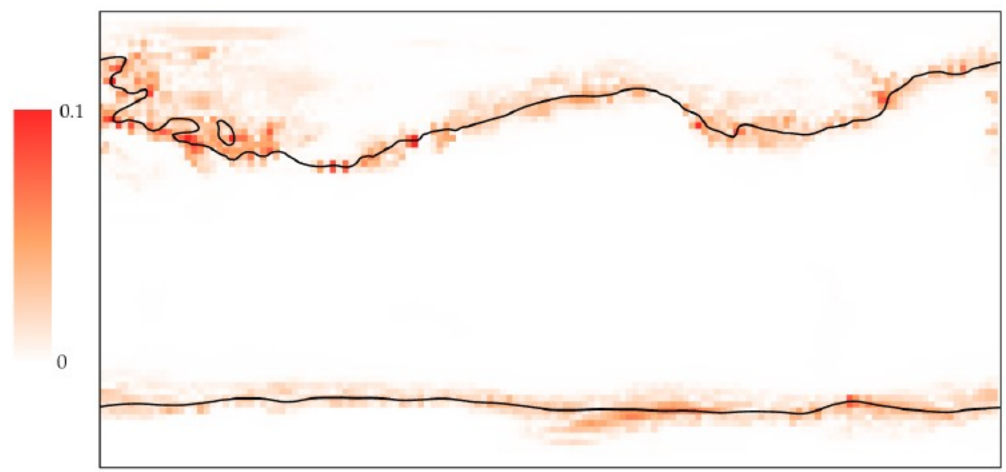
Maximum Edge Approximation



Monte Carlo integration (1000 samples)

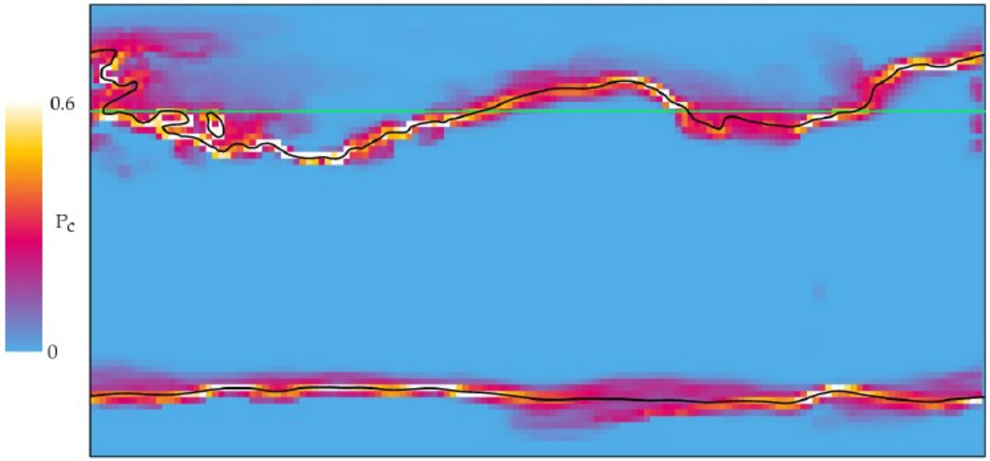


maximum edge approximation

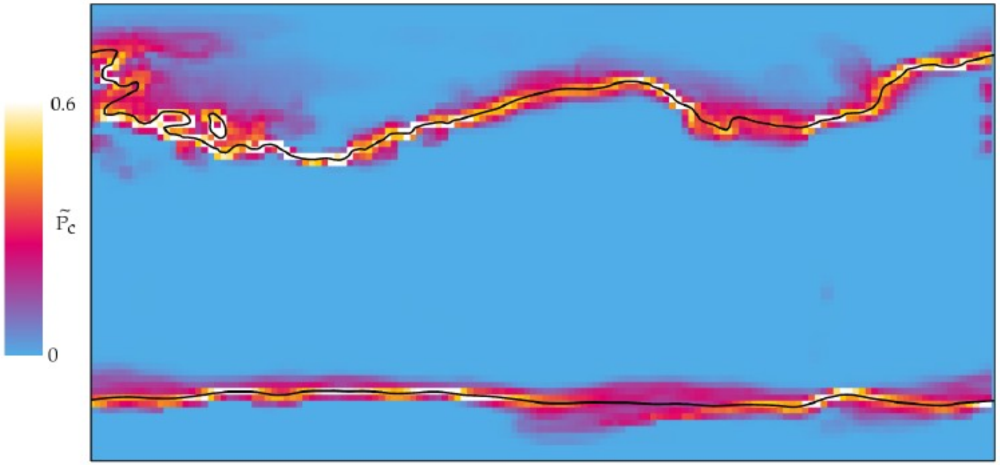


absolute differences

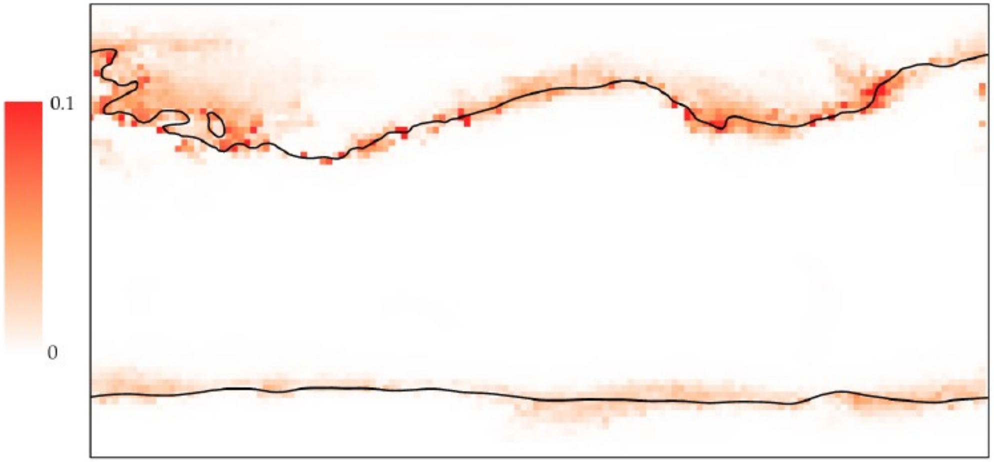
Linked-Pairs Approximation



Monte Carlo integration (1000 samples)

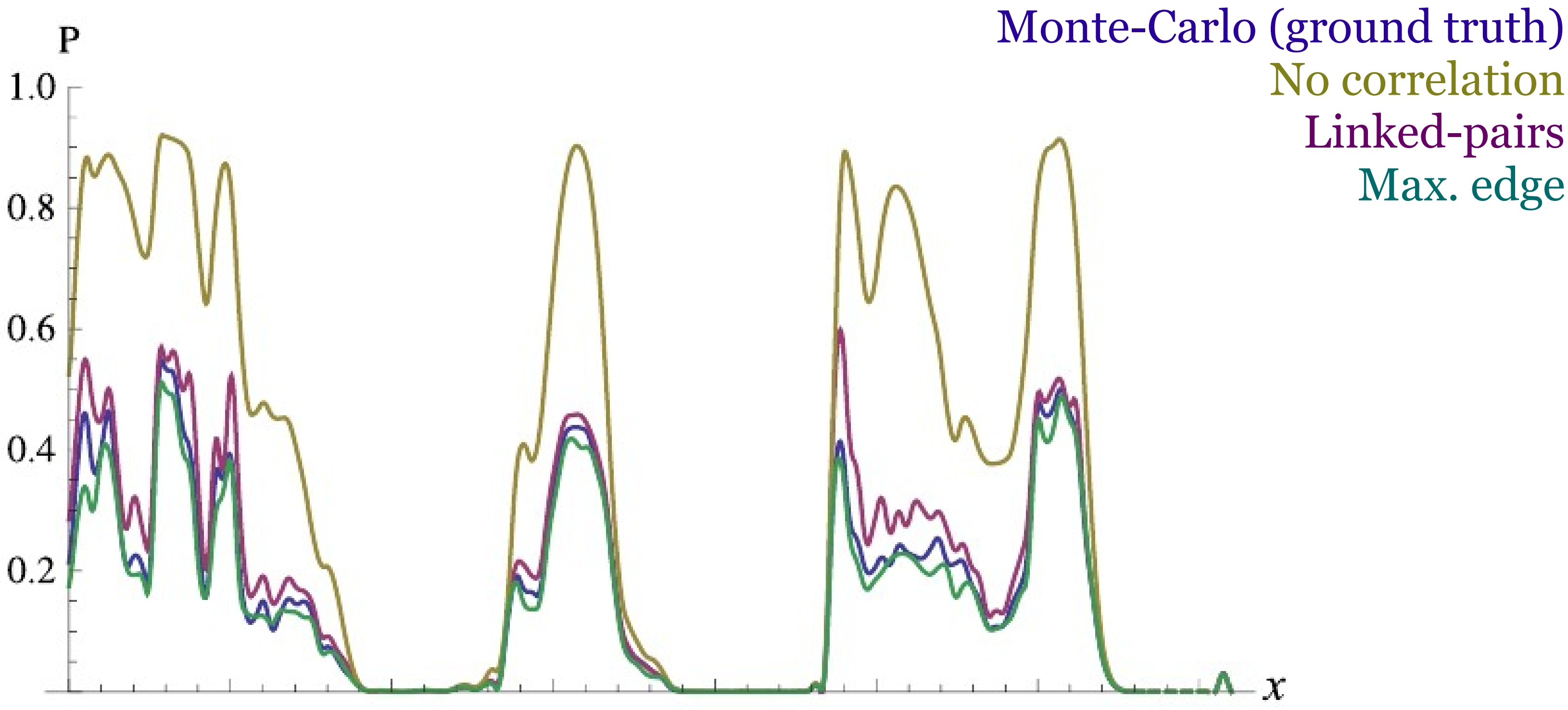


linked-pairs approximation



absolute differences

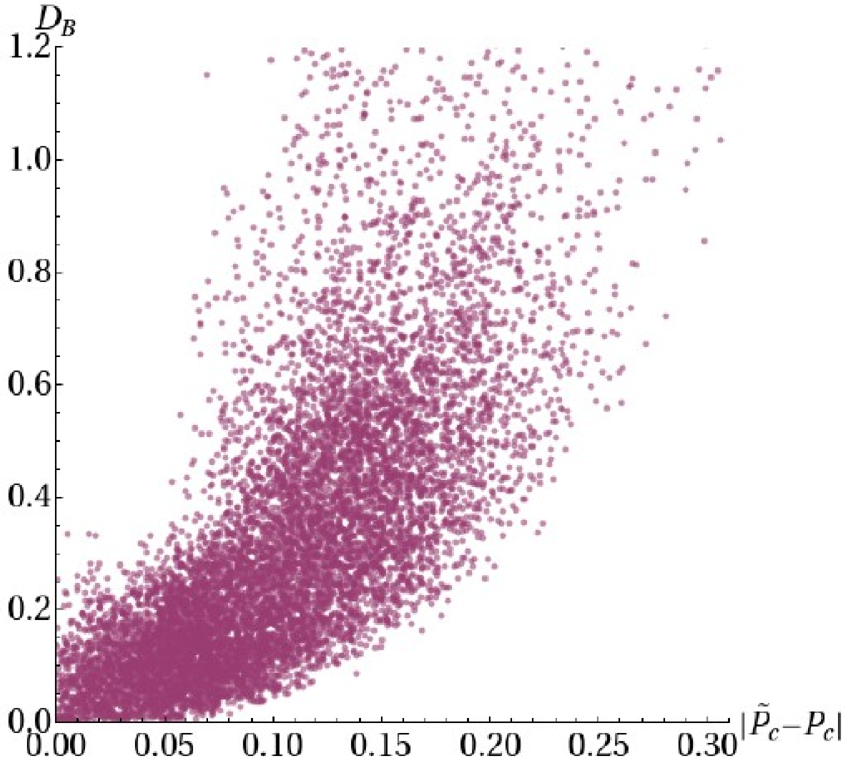
Comparison



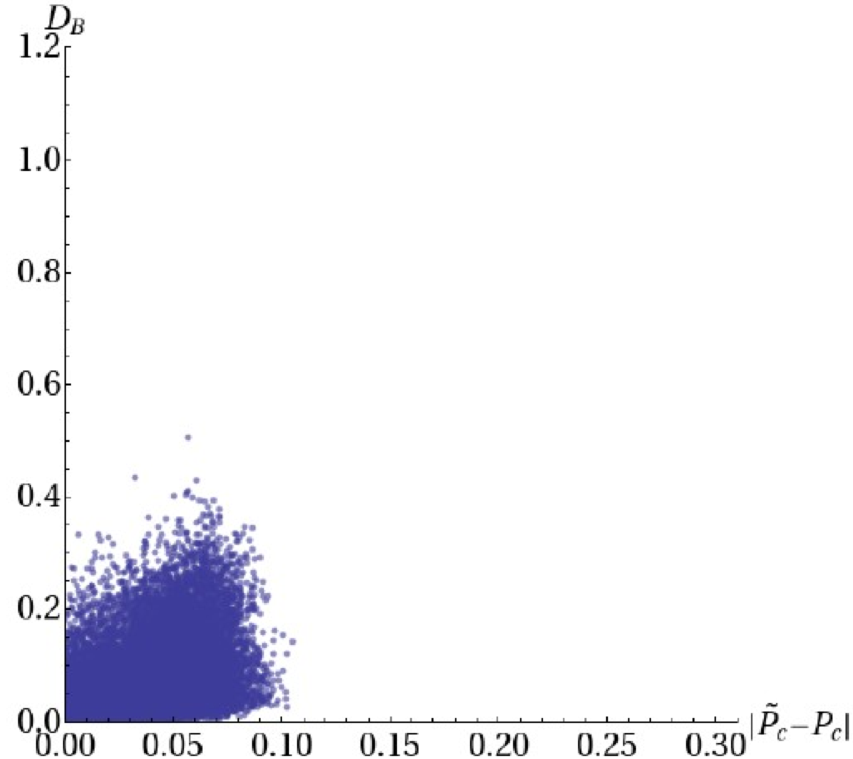
P
1.0
0.8
0.6
0.4
0.2
x
Monte-Carlo (ground truth)
No correlation
Linked-pairs
Max. edge

Optimal Approximate Distribution

What is the impact of the optimization step?

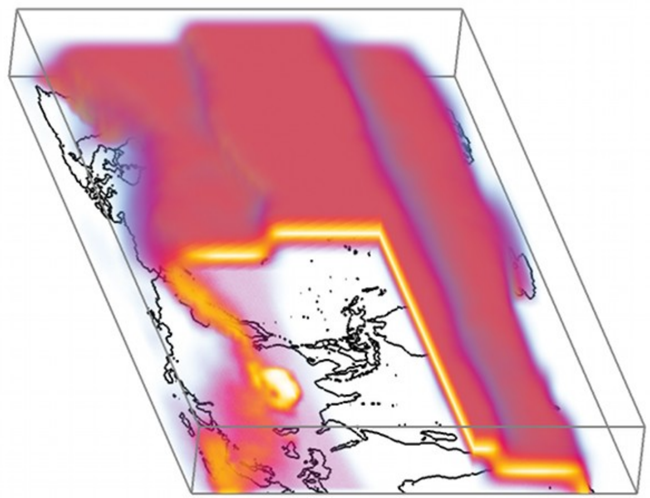


random spanning tree

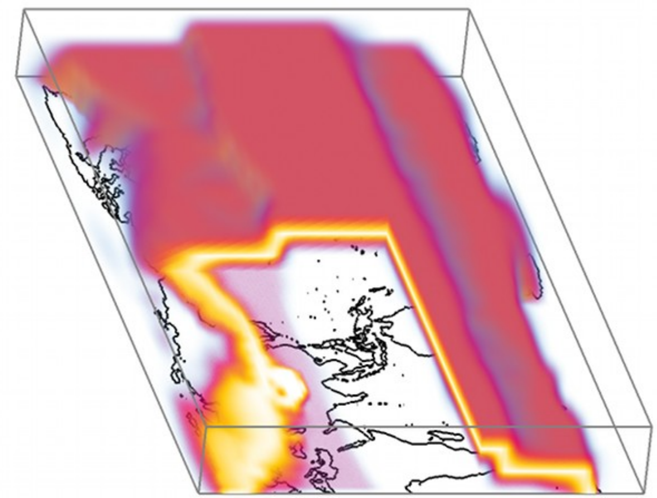


optimal spanning tree

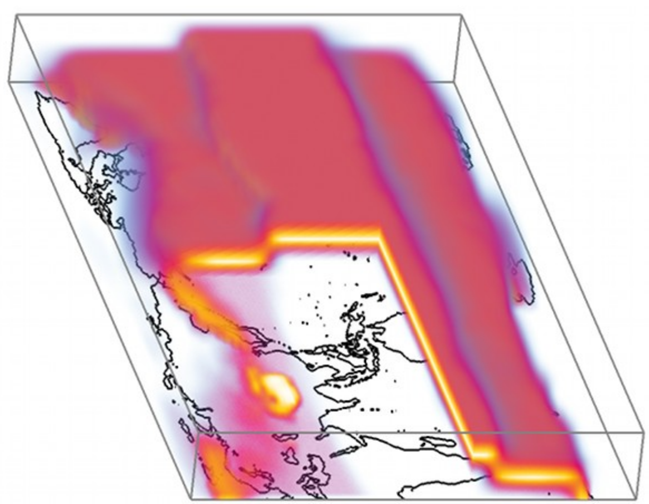
3D Dataset



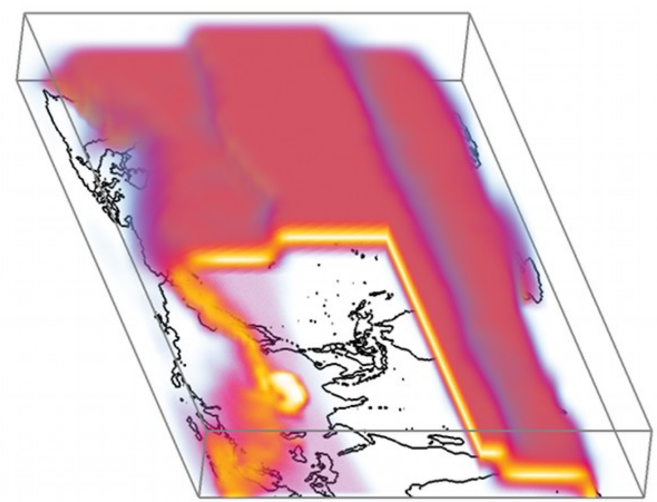
Monte Carlo integration



No correlation considered

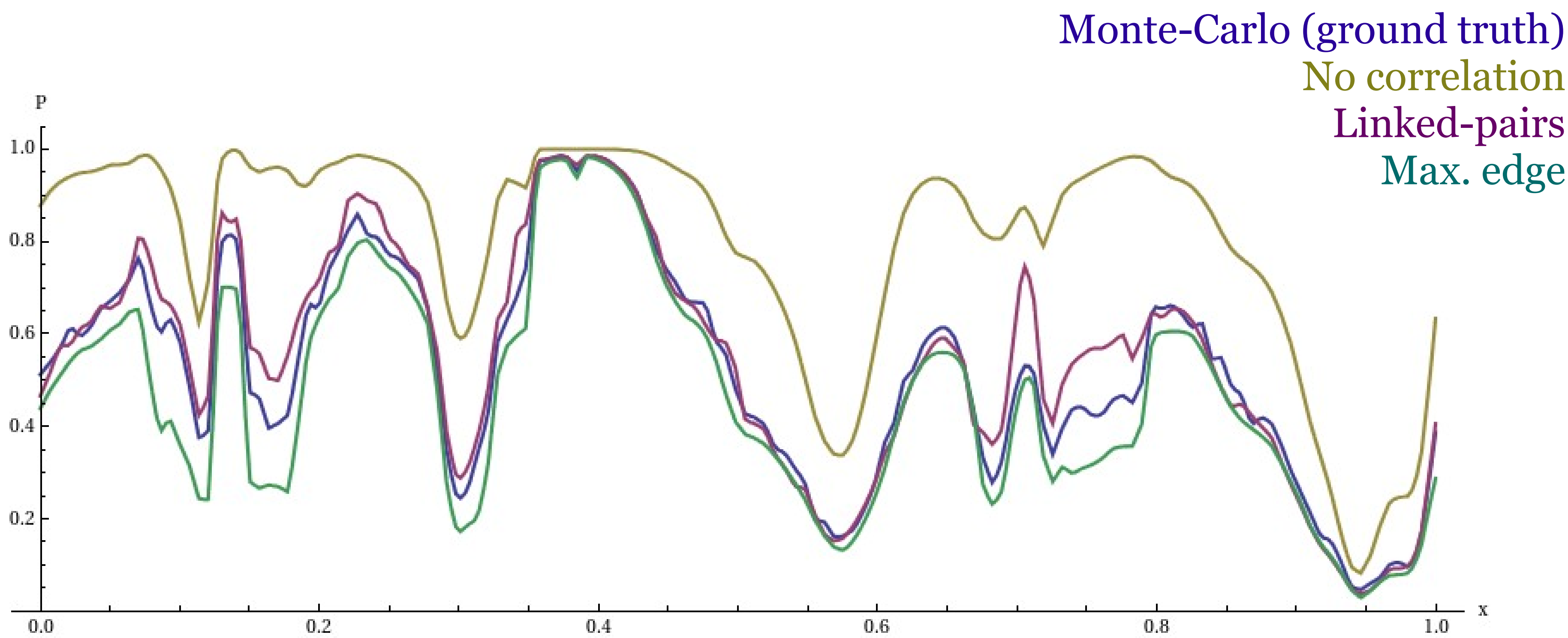


Maximum edge approximation



Linked-pairs approximation

Comparison (3D)



Performance

	Time in seconds*
Monte-Carlo integration 1000 samples/voxel	23
Max.-Edge method	0.17
Linked-Pairs method	0.11

* single-thread, i7, 2.6 GHz

Conclusions

- Approximations allow fast evaluation of level-crossing probabilities using lookup tables
- No Monte-Carlo noise
- Maximum edge method
  - Simple concept
  - Lower bound for true probability
- Linked-pairs method
  - Considers complete random vector
  - Induces approximate distribution
  - Optimization w.r.t. the Bhattacharyya distance significantly increases accuracy
- Resulting visual impressions are close to the MC result

Thank you!