



# Workgroup for applications of PDEs

Masterthesis

# Application of the mixed-FEM approach for the MEG-Problem

Reiner Stubbemann Matr. 428490

Supervisor: Prof. Dr. Christian Engwer Second Supervisor: Prof. Dr. Carsten Wolters

Institute for applied mathematics, University of Münster, Germany

20. November 2021

This thesis revolves around the application of the mixed finite element method (mixed-FEM) on the EEG- and MEG forward problems. In Johannes Vorwerk Dissertation (2016) many results for the projection approach for the EEG forward problem were derived and compared to various different continuous galerkin(ContG) and discontinuous galerkin (DG) source models. He was able to show the ability of the mixed-FEM approach to reduce leakages effects near the skull tissue. The goal of this thesis will be to test the accuracy of the direct source model for the problems stated above, which were not represented in his dissertation. Additionally the solution process will be optimized by deriving the transfer matrices for the direct approach source model and testing it on the MEG forward problem. The numerical tests were based on a combination of up to 10000 dipoles within a three dimensional spherical domain. It was found that the direct approach source model is only an efficient source model for solving the EEG- and MEG forward problems for dipoles with large eccentricity. While having a similar magnitude of accuracy and convergence rate, the additional computational cost and the higher error-rates do not make it the desirable approach compared to the DG-FEM approaches.

# Contents

1	Introduction	1
2	The EEG- and MEG-forward Problem2.1The physiological Background2.2Quasi-Static approximation of the Maxwell-equations2.3The EEG-forward problem2.4The MEG-forward problem2.5Formula for the analytical solutions	<b>3</b> 3 5 6 7 8
3	Mixed-Finite-Element Method	9
	<ul> <li>3.1 Galerkin-Approximations and the finite element method</li></ul>	9 10 11 12 15 16 16 16 18
4	Solvers         4.1       Introduction	<ol> <li>19</li> <li>20</li> <li>20</li> <li>21</li> <li>22</li> <li>22</li> </ol>
5	Implementation         5.1       Software         5.2       Assembling the system matrix         5.3       Transfer matrices         5.3.1       EEG         5.3.2       MEG         5.4       Experiments	<ol> <li>25</li> <li>25</li> <li>27</li> <li>28</li> <li>29</li> <li>30</li> </ol>
6	Evaluation and Conclusion         6.1       Hardware       .	<b>33</b> 33 36 37
Bi	bliography	39

#### CHAPTER 1

## Introduction

Two of the most commonly used modern medical imaging processes are the electroencephalography (EEG) and the magnetoencephalography (MEG). Both methods are used in order to gain information on the neurological activity within the patients head, for example to diagnose epilepsy, sleep disorders, brain damage etc. They are used in different cases due to the EEG often being related to the extracellular ionic currents and the MEG to the intracellular. Mathematically the MEG is based on the solution of the EEG-forward problem. This relation is physically based on the Maxwell-equations.

This means that in order to advance, fitting numerical methods and procedures need to be derived. The most commonly used numerical approaches are continuous finite element methods (CG-FEM). Using those and different discretizations for the source within the head the electric potential on the heads surface can be calculated. While this only affects the forward problem, the inverse problem can then be solved using for an example a dipole fit procedure. In his dissertation Johannes Vorwerk proofed that the mixed-FEM approach, analogous to the DG-FEM approach, can guarantee the conservation of charge. Further experiments proofed for example that using mixed-FEM the modeling leakage effects in the skull-tissue could be avoided. The results Vorwerk presented were however mostly referring to only one of the approaches he derived to discretice the source, namely the projected approach. The goal of this thesis is therefore to gather more data on the direct approach that was also derived, and to optimize the calculation for multiple dipoles using a transfer matrix approach for the EEG forward problem.

Beforehand the partial differential equation (PDE) for the mixed-approach has to be derived using the biological and physical conditions and requirements. Once the equation is derived, the corresponding weak formulation will be calculated which then leads to a numerically solvable matrix equation. While basic numerical solvers could be used to solve this equation, its unusual saddle-point structure requires a different solving-approach to enable realistic computation costs. Finally, once the matrix-system and the solver structure have been established, the test-environment can be explained and used to gather data on the mixed-approach which will then be evaluated in the final steps.



Figure 1.1: Visualization of model geometry (left column), current direction and strength for partial integration approach and CG-FEM (middle column) and DG-FEM (right column) for models seg 2 res 2 r82 (top row), seg 2 res 2 r83 (middle row), and seg 2 res 2 r84 (bottom row). The left column shows the model geometry, interior to exterior from bottom left to top right, brain in white, CSF, skull and skin in increasingly dark gray, and air in white. Dark gray lines mark compartment boundaries. In the middle and right columns, the large turquoise cone represents the dipole source. The small and normalized grey cones show the directions of the current flow and, for elements belonging to skull and skin compartments, the coloring indicates the current strength. For each model the color scale is kept constant for both approaches. src: Johannes Vorwerk (2016)

#### CHAPTER 2

## The EEG- and MEG-forward Problem

To gain an overview this chapter is restricted to providing the fundamental information necessary to understand the biological and physical background behind the electroencephalography-(EEG) and magnetoencephalography (MEG) problems. These are based on the electromagnetical reactions inside the human head, which can be measured on either the heads surface - using multiple electrodes - or - by using multiple magnetic coils - measuring the emerging magnetic flux surrounding the head. The correlation between these measurements can be described using the Maxwell equations which therefore can be used to derive the partial differential equations necessary to formalize these forward problems and therefore get the mathematical requirements for solving them. Finally an equation for the analytical solution on domains consisting only of multilayered spheres will be given but not derived, in order to be able to validate first solutions calculated without exceeding the extend of this thesis. Most of the theory and structure of this work is based on the dissertation of Johannes Vorwerk (2016), therefore it should be reviewed if further information are required or to check on the theory of the usual approaches beyond the mixed-approach. At this point is it also necessary to explain why it is sufficient to derive the respective forward solution, when the solution of their inverse problems is actually of medical and scientific interest. Using the dipole fix procedure or the Landweber-Iteration these solutions can be approximated by solving the forward problems in each step of the iteration. The idea for the Landweber-Iteration is that instead of solving the inverse problem,  $x = A^{-1}y$  for an operator A, which is not well defined, meaning that it is discontinuous for the input data y, one stabilizes the solution by solving  $\min_x ||Ax - y||_2^2$ . The algorithm then looks like  $x_{k+1} = x_k - \omega A^*(Ax_k - y)$ . By setting  $f(x) = ||Ax - y||_2^2/2$  one can rewrite the equation above as  $x_{k+1} = x_k - \omega \nabla f(x_k)$  which therefore is a special case of the gradient descend. This has to be done because due to fact that small errors in the input data y can be heavily amplified due to the discontinous nature of the inverse operator for the problem at hand. With this setup one can proceed as mentioned above.

#### 2.1 The physiological Background

In order to understand the emerging electromagnetical, neurological reactions, the basic composition of the brain and its functionality will be depicted in the following. The brain consists of around  $10^{11}$  to  $10^{12}$  neurons, which communicate using specific electrochemical signals. The neuron consists of up to 7000 dendrites which are connected to the soma and its respective, unique axon, a connection to the synapse which, itself once again, connects to the dendrites of different neurons (Drachman, 2005).

The electric potential differences are caused by an ion flow within and between multiple

neurons. Due to changes of the conductivity, positive and negative charged ions flow in different manners causing higher or lower local electric potentials which then, as implied by the Maxwell-equations, lead to differences in the magnetic field.

There are three different bio-chemical reactions causing the measurable potentials: The pre-synaptic -, the post-synaptic- and the action potential.

As part of cell-to-cell communication a summation of all incoming potentials at the axon hillock of a neuron is started. If the summated potential exceeds a certain threshold a sudden change of the membrane potential, perceptible as a rapid rise followed by a fall, occurs, which is called action potential. This action potential is then transported along the axon and when it reaches the synaptic bouton, current-reliant calcium-channels are opened and ca<sup>2+</sup>-ions diffuse into the synapse. As a consequence of that, vesicles filled with neurotransmitters fuse with the pre-synaptic membrane and release the respective transmitters into the synaptic gap. The neurotransmitters diffuse to the postsynaptic membrane and bind to specific receptors of ligand-controlled ion channels. The influx of ions leads to a post-synaptic potential which, depending on the type of synapse, can be excitatory or inhibitory. In case of an excitatory post-synaptic potential the potential changes may add up again and could cause an action potential on the axon hillock in the post-synaptic neuron, which means that the signal is passed on. The frequency and strength of the action potential depends on the concentration of neurotransmitters in the synaptic gap. A high frequency that arrives at the membrane of the synaptic bouton also results in a high transmitter concentration in the synaptic gap and a correspondingly higher frequency of action potentials on the postsynaptic membrane. Due to the short duration of the action potential (0.5-2 ms) and the fact that the signal is not synchronized and the signal is dominated by quadrupole terms, the signal itself is not exploitable for the EEG measurements (Plonsey, 2005).

The post-synaptic potential that remains due to the diffused neuro-transmitters however has a length of 15 - 20 ms along similar oriented neurons, corresponding to a few square millimeters of the cortex surface, which then leads to a dipolar electromagnetic field, which then again can be measured.



Figure 2.1: Simplified structure of the neuron where  $x_1, ..., x_n$  denote the input signal and  $y_1, ..., y_n$  the output signal

#### 2.2 Quasi-Static approximation of the Maxwell-equations

In order to proceed to the formulation of the partial differential equation for the EEGforward problem, a quasi-static approximation of the Maxwell-equations is needed. Those equations describe the relations between magnetic - and electric fields. The reason why this formulation will be used, will be explained in this chapter as well. When it is assumed that the permeability  $\mu$  of the heads tissues is equal to that of free space, denoted as ( $\mu_0$ ), the Maxwell-equations read as

$$\nabla \cdot E = \frac{\rho}{\epsilon_r \epsilon_0},\tag{2.1}$$

$$\nabla \cdot B = 0, \tag{2.2}$$

$$\nabla \times E = -\frac{\partial B}{\partial t},\tag{2.3}$$

$$\nabla \times B = \mu_0 j + \mu_0 \epsilon_r \epsilon_0 \frac{\partial E}{\partial t}, \qquad (2.4)$$

where E and B denote the electric- and magnetic-field, respectively,  $\rho$  the electric density charge, j the current density,  $\epsilon_r$  the relative permittivity and lastly  $\epsilon_0$  the permittivity of free space.

In the following, two things need to be taken into consideration. First, using Ohm's law, the electric charge density can be written as the product of the underlying conductivity  $\sigma$ and the electric field E, i.e.  $j = \sigma E$ . Second, the assumption is made that a harmonic time dependency for the electric field with an angular frequency  $\omega$  exists, meaning that  $E(t) = E_0 e^{-i\omega t}$ . Here no generality is lost due to the fact the Fourier decomposition could be applied, without changing the results. Proceeding by applying the steps mentioned above on equation (2.4) provides

$$\nabla \times B = \mu_0 (\sigma - \epsilon_r \epsilon_0 i\omega) E_0 e^{-i\omega t}.$$
(2.5)

The requirement for using the quasi-static approximation in (2.4) is that the time-dependent term is small in relation to the non-dependent part, i.e.  $|\epsilon_r \epsilon_0/\sigma| \ll 1$ . Using the magnitudes  $\sigma = 0.3S/m$ ,  $\epsilon_r = 10^5$  (Vorwerk, 2016) and a frequency  $f = \omega/2\pi \approx 100Hz$  (Hämäläinen et al., 1993), we find that  $|\epsilon_r \epsilon_0/\sigma| \approx 2 \cdot 10^{-3} \ll 1$ . Additionally, applying (2.3) on (2.5) leads to

$$\nabla \times (\nabla \times E) \stackrel{(2.3)}{=} -\frac{\partial}{\partial t} (\nabla \times B)$$
(2.6)

$$\stackrel{(2.5)}{=} \mu_0 (i\sigma\omega + \epsilon_r \epsilon_0 \omega^2) E_0 e^{-i\omega t}.$$
(2.7)

As stated in (Vorwerk, 2016) the solutions of this equation have a characteristic wavelength of 65m and can therefore be neglected when looking at a much smaller scale like a human head.

This results in the formulation

$$\nabla \times E = 0, \tag{2.8}$$

$$\nabla \cdot E = \frac{\rho}{\epsilon_0 \epsilon_r} \tag{2.9}$$

for the electric-field E and

$$\nabla \times B = \mu_0 j, \tag{2.10}$$

$$\nabla \cdot B = 0 \tag{2.11}$$

for the magnetic-field B.

#### 2.3 The EEG-forward problem

The derivation of the partial differential equation is based on the existence of an scalar, electric potential u for which

$$E = -\nabla u. \tag{2.12}$$

This equation holds due to the fact that E is a gradient field, as implied by (2.8). It is necessary to understand the current j as a sum of the primary current  $j^p$ , which relates to the neural brain activity, and the volume or return current  $j^v = \sigma(x)E(x)$  (Ohm's law), that is

$$j(x) = j^{p}(x) + j^{v}(x), \text{ for } x \in \mathbb{R}^{3}.$$
 (2.13)

Applying the equation (2.12) leads to

$$j(x) = j^{p}(x) - \sigma(x)\nabla u(x).$$
(2.14)

When using the divergence on (2.10) one additionally finds that

$$\nabla \cdot j = 0 \tag{2.15}$$

Now combining (2.15), (2.14) and assuming homogeneous Neumann boundary conditions one derives the partial differential equation for the EEG-forward problem as

$$j + \sigma \nabla u = j^p, \tag{2.16}$$

$$\nabla \cdot j = 0, \tag{2.17}$$

$$\langle j, n \rangle = 0. \tag{2.18}$$

At this point it should be noted that one could eliminate j. Doing this, the non-mixed formulation of the EEG-forward problem, which is used in CG-, DG- and Cut-FEM approaches, emerges:

$$\nabla \cdot (\sigma \nabla u) = \nabla \cdot j^p, \qquad (2.19)$$

$$\sigma \nabla u \cdot n = 0. \tag{2.20}$$

In both equations, the source  $j^p$  can be written as a composition of current dipoles, i.e.  $j^p(x) = M\delta_{x_0}(x)$ , where M denotes the dipole-moment and  $\delta_{x_0}$  denotes a Dirac distribution in  $x_0 \in \mathbb{R}^n$ . This n depends on the domain  $\Omega$ , which is either part of  $\mathbb{R}^3$  in case of a realistic-or 3d-headmodel, or  $\mathbb{R}^2$  in special test-cases.

#### 2.4 The MEG-forward problem

After solving the EEG-forward problems (2.16)-(2.18) or (2.19)-(2.20) for either (j, u) or u, respectively, the solution can be used to solve the MEG-forward problem. The magnetic-flux  $\Phi$  can be calculated using the magnetic field B as

$$\Phi = \int_{S} B \cdot ds, \tag{2.21}$$

where S denotes the sensor-surface.

The existence of a magnetic vector potential  $\tilde{B}$ , which satisfies the Coulomb gauge  $(\nabla \cdot \tilde{B} = 0)$ , such that  $B = \nabla \times \tilde{B}$ , is implied by (2.11) (Piastra, 2019), resulting in:

$$\mu_0 j = \nabla \times B$$
  
=  $\nabla \times (\nabla \times \tilde{B})$   
=  $\nabla (\nabla \cdot \tilde{B}) - \nabla^2 \tilde{B}$   
=  $-\nabla^2 \tilde{B}$ , (2.22)

where (2.22) is a Poisson equation with solution

$$\tilde{B}(r) = \frac{\mu_0}{4\pi} \int_{\Omega} \frac{j(r')}{|r-r'|} d^3 r'.$$
(2.23)

Using this it can be concluded that

$$B(r) = \nabla \times \left(\frac{\mu_0}{4\pi} \int_{\Omega} \frac{j(r')}{|r-r'|} d^3 r'\right)$$
  
=  $\frac{\mu_0}{4\pi} \int_{\Omega} \nabla \times \left(\frac{j(r')}{|r-r'|}\right) d^3 r'$   
=  $\frac{\mu_0}{4\pi} \int_{\Omega} j(r') \times \frac{r-r'}{|r-r'|^3} d^3 r'.$  (2.24)

If the mixed-FEM approach is used, in which case the flux j was calculated when solving the EEG-forward problem, (2.24) gives us the formula to calculate the magnetic field B. In case of the other approaches a few additional steps have to be taken into account. Using (2.13), the equation above can be written as

$$B(r) = \frac{\mu_0}{4\pi} \int_{\Omega} (j^p(r') + j^v(r')) \times \frac{r - r'}{|r - r'|^3} d^3r'$$
(2.25)

$$= \frac{\mu_0}{4\pi} \int_{\Omega} (M\delta_{x_0}(r') + (-\sigma\nabla u)(r')) \times \frac{r - r'}{|r - r'|^3} d^3r'$$
(2.26)

$$= \frac{\mu_0}{4\pi} M \times \frac{r - x_0}{|r - x_0|^3} - \frac{\mu_0}{4\pi} \int_{\Omega} \sigma \nabla u(r') \times \frac{r - r'}{|r - r'|^3} d^3 r'.$$
(2.27)

The first part of the equation is known as primary *B*-field  $B^p = \frac{\mu_0}{4\pi}M \times \frac{r-x_0}{|r-x_0|^3}$  and can be calculated analytically.  $B^s = -\frac{\mu_0}{4\pi}\int_{\Omega}\sigma\nabla u(r') \times \frac{r-r'}{|r-r'|^3}d^3r$  denotes the secondary *B*-field and has to be calculated numerically using the solution u of the EEG-forward problem.

#### 2.5 Formula for the analytical solutions

This chapter will only give a brief expression of the analytical solutions. These are taken from Piastra (2019). More information on these are given by (Sarvas, 1987) and (Munck and Peters, 1993).

Under the assumption of simple geometries, a (quasi)-analytical solution for the MEGproblem can be stated. For a multi-layer, spherical, homogeneous model this solution is given by

$$B(r) = \frac{\mu_0}{4\pi F^2} (FM \times r_0 - M \times r_0 \cdot r\nabla F), \qquad (2.28)$$

where  $a = r - x_0$ ,  $\alpha = ||a||$ ,  $\tilde{r} = ||r||$ ,  $F = \alpha(\tilde{r}\alpha + \tilde{r}^2 - r \cdot x_0)$  and  $\nabla F = (\tilde{r}^{-1}\alpha^2 + \alpha^{-1}a \cdot r + 2\alpha + 2\tilde{r})r - (\alpha + 2\tilde{r} + \alpha^{-1}a \cdot r)x_0$  (Sarvas, 1987).

For the EEG-problem, the analytical solution can only be given for a model consisting of N overlapping centered spheres with radii  $r_0 < r_1 < \cdots < r_N$  and constant radial and tangential conductivities in each sphere. In the following, it is assumed that the dipole is within a deeper sphere than the electrodes at position  $y_i$ , for s different sensors, with radial coordinate  $y_i^r$ . When  $x^r$  then denotes the radial coordinate of a vector x the potential u at  $y_i$  evoked by a source  $j^p = M\delta_x$  is given by

$$u(x, p, y_i) = \frac{1}{4\pi} \left( p, \frac{S_0}{\gamma_i^r} y_i + \left( \frac{S_1}{x^r} - \cos \omega_{x, y_i} \frac{S_0}{x^r} \right) x \right)$$
(2.29)

where  $\omega_{x,y_i} = \arccos \langle \frac{x}{\|x\|_2}, \frac{y_i}{\|y_i\|_2} \rangle$  denotes the radial distance between x and  $y_i$ . Additionally,  $S_0$  and  $S_1$  are given by

$$S_{0} = \frac{F_{0}}{x^{r}} \frac{\Lambda}{(1 - 2\Lambda \cos \omega_{x,y_{i}} + \Lambda^{2})^{3/2}} + \frac{1}{x^{r}} \sum_{n=1}^{\infty} [(2n+1)R_{n}(x^{r}, y_{i}^{r}) - F_{0}\Lambda^{n}]P_{n}'(\cos \omega_{x,y_{i}}),$$
  

$$S_{1} = F_{1} \frac{\Lambda \cos \omega_{x,y_{i}} - \Lambda^{2}}{(1 - 2\Lambda \cos \omega_{x,y_{i}} + \Lambda^{2})^{3/2}} + \sum_{n=1}^{\infty} [(2n+1)R_{n}'(x^{r}, y_{i}^{r}) - F_{1}n\Lambda^{n}]P_{n}(\cos \omega_{x,y_{i}}),$$

where  $P_n$  and  $P'_n$  are the Legendre polynomial and its derivative and the  $R_i$  and  $R'_i$  are the respective coefficients. The remaining constants can be found in the work of Munck and Peters (1993). As the sum can not be completely computed, the calculation of the sum is stopped once a certain condition is met. This condition will not be further evaluated here but it was shown by (de Munck and Peters, 1993).

### Mixed-Finite-Element Method

Under the most used methods for solving partial differential equations are finite element methods (FEM), which are special cases of Galerkin approximations. Therefore, before deriving the discretizations for the EEG- and MEG-problem, the fundamentals of the Galerkin-approach and the general finite element method setup are briefly explained. The notation and structure of this chapter are based on the script for the lecture of numerics of partial differential equations by Professor Dr. Wirth (2015/2016). For showing the existence and uniqueness of the solution of the later derived equation, the work from Vorwerk (2016) will be used, which took the proofs from Braess (2007).

#### 3.1 Galerkin-Approximations and the finite element method

Assume that V denotes a finite-dimensional subspace of the solutions function space F, the function space of functions  $u : \Omega \to \mathbb{R}$ , with basis  $\phi_1, ..., \phi_k$ . While usually, when solving a partial differential equation, one minimizes the respective energy-functional E for all functions found in F, one now minimizes the functional only for the above mentioned subspace V. This means instead of calculating  $u = \arg \min_{v \in F} E(v)$ , one now solves  $\tilde{u} =$  $\sum_{i=1}^{k} u_i \phi_i = \arg \min_{v \in V} E(v)$ . In this case  $\tilde{u}$  is called a Galerkin-approximation of u. If the  $\phi_i$  only have local support, i.e.  $\phi_i(x) = 0$  for  $x \notin supp(\phi_i)$ , the special case of finite elements emerge. For these, different possible subspaces can be used depending on the discretization used. In order to be able to describe necessary basis-functions one therefore needs a definition describing how the domain is discretized.

**Definition 1** Let  $\Omega$  be a subset of  $\mathbb{R}^n$ . We call T a admissible decomposition of  $\Omega$  when:

1) 
$$\overline{\Omega} = \bigcup_{T_i \in T} T_i, \tag{3.1}$$

2) 
$$T_i \neq \emptyset, \forall i$$
 (3.2)

3) for 
$$i \neq j$$
,  $T_i \cap T_j = \partial T_i \cap \partial T_j$ , and  $codim(T_i \cap T_j, \Omega) > 0$  (3.3)

We generally assume, that the decompositions we are using are shape-regular. Especially, for the sake of simplicity, decompositions using hexaedric elements  $T_i$  will be used in the following work. It should be mentioned that the entire theory works analogous for tetraedric decompositions.

#### 3.2 Variational formulation of the EEG-Problem

Before the weak forms of the EEG- and MEG-problem are derived, the necessary subspaces of the potential- and flux-spaces are needed. For this, two spaces need to be defined beforehand:

$$W^{k,p}(T) := \{ v \in L^2(\Omega) : v | _{T_i} \in W^{k,p}(T_i) \mid \forall T_i \in T \}$$
(3.4)

$$H^{k}(T) := W^{k,2}(T), (3.5)$$

which are, unlike their local variants, potentially not differentiable between different elements of the decomposition, for k > 0.

Now that the necessary spaces for weak solutions were presented, the EEG-problem (2.16-18) should be restated:

$$j + \sigma \nabla u = j^p, \tag{3.6}$$

$$\nabla \cdot j = 0, \tag{3.7}$$

$$\langle j, n \rangle = 0. \tag{3.8}$$

First, it should be noted that the first equation has to be multiplied with the inverse tensor  $\sigma^{-1}$ , to ensure that the resulting system will be symmetric. Then, using the general approach for deriving the weak formulation, the system (3.6) is integrated and multiplied with a vector-valued test function.

$$\int_{\Omega} \langle \sigma^{-1} j + \nabla u, \phi \rangle dx = \int_{\Omega} \sigma^{-1} j^p \cdot \phi dx \tag{3.9}$$

$$\Leftrightarrow \int_{\Omega} <\sigma^{-1}j, \phi> + <\nabla u, \phi> dx = \int_{\Omega} \sigma^{-1}j^{p} \cdot \phi dx \tag{3.10}$$

$$\Leftrightarrow \int_{\Omega} <\sigma^{-1}j, \phi> + < u, \nabla \cdot \phi > dx = \int_{\Omega} \sigma^{-1}j^{p} \cdot \phi dx \tag{3.11}$$

$$\Leftrightarrow \int_{\Omega} <\sigma^{-1}j, \phi > dx + \int_{\Omega} < u, \nabla \cdot \phi > dx = \int_{\Omega} \sigma^{-1}j^{p} \cdot \phi dx$$
(3.12)

This leads to the bilinear forms

$$a(w,v) = \langle \sigma^{-1}w, v \rangle_{L^2(\Omega)^3}$$
(3.13)

$$b(w,k) = \langle \nabla \cdot w, k \rangle_{L^2(\Omega)} \tag{3.14}$$

$$l(v) = \langle \sigma^{-1} j^p, v \rangle_{L^2(\Omega)^3} \tag{3.15}$$

where w and v are vector-valued  $H_*(div, \Omega)$  functions and k is an  $L^2(\Omega)$ -function. The system (2.15-16) defines the weak-problem that needs to be solved for  $(j, u) \in H_*(div, \Omega) \times L^2(\Omega)$ , where  $H_*(div, \Omega) = \{q \in L^2(\Omega)^3 : \nabla \cdot q \in L^2(\Omega) \land \langle q |_{\partial\Omega}, n \rangle = 0\}$  to find the solution to the EEG-forward problem:

$$a(w, v) + b(w, k) = l(w)$$
 (3.16)

$$b(w,l) = 0. (3.17)$$

#### 3.3 Existence and uniqueness of a weak solution

This section will complete the first tasks at hand: proving the existence and uniqueness for the solution of the equations (2.16-17).

**Theorem 1** For every  $l \in H^1(\Omega)'$ , the problem (2.16-17) admits a unique solution  $(j, u) \in H_*(div, \Omega) \times L^2(\Omega) = V \times Q$ .

Proof: This proof follows the steps analogous to (Johannes Vorwerk, 2016, S.141 ff.) According to Braess (2007) it is sufficient to proof that a from (2.13) is  $V_0$ -elliptic and that b from (2.14) fulfills the following inf-sup condition

it exists 
$$\beta > 0$$
 such that (3.18)

$$\inf_{q \in Q/\{0\}} \sup_{v \in V/\{0\}} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \ge \beta.$$
(3.19)

First:

We recall that with  $V_0 := \{v \in V : (\nabla \cdot v, q) = 0, \forall q \in Q\}$  it can be shown that the form a is bounded and therefore continuous.

For  $v \in V_0 \subset H(div, \Omega)$  it holds that  $\nabla \cdot v \in Q = L^2(\Omega)$  and therefore  $\|\nabla \cdot v\|_{L^2(\Omega)} = 0$ leading to

$$a(v,v) \ge \min_{x \in \Omega} \sigma^{-1}(x) \|v\|_{L^2(\Omega)^3}^2 = \alpha \|v\|_{L^2(\Omega)^3}$$
(3.20)

$$= \alpha(\|v\|_{L^{2}(\Omega)^{3}}^{2} + \|\nabla \cdot v\|_{L^{2}(\Omega)}^{2}) = \alpha\|v\|_{H(div,\Omega)}^{2}.$$
 (3.21)

Therefore a is  $V_0$ -elliptic.

Second:

For  $0 \neq q \in L^2(\Omega)$ . Using that  $C_0^{\infty}(\Omega)$  is dense in  $L^2(\Omega)$ , there exists  $\tilde{q} \in C_0^{\infty}(\Omega)$  such that

$$\|q - \tilde{q}\|_{L^2(\Omega)}^2 \le \frac{1}{2} \|q\|_{L^2(\Omega)}^2.$$
(3.22)

Defining a  $v \in L^2(\Omega)^3$  by setting its first component as

$$v_1(x) = v_1(x_1, x_2, x_3) := \int_{-\infty}^{x_1} \tilde{q}(t, x_2, x_3) dt$$
(3.23)

and  $v_2 = v_3 = 0$ , one finds that

$$\nabla \cdot v = \frac{\partial v_1}{\partial x_1} = \tilde{q}.$$
(3.24)

Now using the Young- and the parallelogram inequality, as well as (5.13), one finds that

$$b(v,q) \ge \frac{1}{2} \|\tilde{q}\|_{L^{2}(\Omega)} \|q\|_{L^{2}(\Omega)}$$
(3.25)

Additionally one can use Friedrich's inequality

$$\|v\|_{L^{2}(\Omega)^{3}} \le c \|\nabla \cdot v\|_{L^{2}(\Omega)}, \qquad (3.26)$$

$$\|v\|_{H(div,\Omega)}^2 \le (1+c^2) \|\tilde{q}\|_{L^2(\Omega)}^2$$
(3.27)

$$\frac{b(v,q)}{\|v\|_{H(div,\Omega)}\|q\|_{L^{2}(\Omega)}} \ge \frac{1}{2} \frac{\|\tilde{q}\|_{L^{2}(\Omega)}\|q\|_{L^{2}(\Omega)}}{\|v\|_{H(div,\Omega)}\|q\|_{L^{2}\Omega}}$$
(3.28)

$$\geq \frac{c'}{2} \frac{\|\tilde{q}\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}}{\|\tilde{q}\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}} \geq \frac{c'}{2}$$
(3.29)

where both requirements are fulfilled and we can find a unique solution  $(j, u) \in H(div, \Omega) \times L^2(\Omega)$ . It remains to prove that  $u \in H^1(\Omega)$ . Since  $C_0^{\infty}(\Omega) \subset H(div, \Omega)$  we find that for any  $\phi \in C_0^{\infty}(\Omega)^3$  and solution u, it holds

$$\langle \nabla \cdot \phi, u \rangle_{L^2(\Omega)} = b(\phi, u) \tag{3.30}$$

$$= -a(j,\phi) + l(\phi) \tag{3.31}$$

$$= -\langle \sigma^{-1}j, \phi \rangle_{L^2(\Omega)^3} + \langle \sigma^{-1}j^p, \phi \rangle_{L^2(\Omega)^3} x$$
(3.32)

$$= \langle \sigma^{-1}(j^p - j), \phi \rangle_{L^2(\Omega)^3}, \text{ for all } \phi \in C_0^\infty(\Omega)^3,$$
(3.33)

where we can use  $\phi_i = e_i \tilde{\phi}$ , for i = 1, 2, 3, leading to

$$\langle \partial \tilde{\phi} / \partial x_i, u \rangle_{L^2(\Omega)} = \langle \sigma^{-1}(j^p - j), \tilde{\phi} \rangle_{L^2\Omega}.$$
 (3.34)

Since this equation holds true for all  $\phi \in C_0^{\infty}$  and i = 1, 2, 3, we find that  $\partial u / \partial x_i = (\sigma^{-1}(j^p - j))_i$ , which then again leads to  $u \in H^1(\Omega)$  with  $\nabla u = \sigma^{-1}(j^p - j)$ .  $\Box$ 

#### 3.4 Discretization of the potential - and flux spaces

Now the two respective subspaces for  $H(div, \Omega)$  and  $L^2(\Omega)$  need to be chosen and validated. For this, the following theorem from Johannes Vorwerk (2016, p. 141) as well as Brezzi and Fortin (1991) is required:

**Theorem 2 (Fortin's criterion (1991))** Let the bilinearform  $b: V \times Q \to \mathbb{R}$  fulfill the inf-sup condition from (2.19). Further, assume that for the subspaces  $V_h$  and  $Q_h$  there exists a bounded, linear projection  $\Pi_h: V \to V_h$ , such that

$$b(v - \Pi_h(v), q_h) = 0 \text{ for all } q_h \in Q_h.$$

$$(3.35)$$

If  $||\Pi_h|| < C$  for some constant C independent of h, also  $V_h$  and  $Q_h$  fulfill the inf-sup condition (2.19).

Proof: For  $q_h \in Q_h$ , we have

$$\beta \|q_h\|_Q \le \sup_{v \in V/\{0\}} \frac{b(v, q_h)}{\|v\|_V}$$
(3.36)

$$= \sup_{v \in V/\{0\}} \frac{b(\Pi_h v, q_h)}{\|v\|_V}$$
(3.37)

$$= c \sup_{v \in V/\{0\}} \frac{b(\Pi_h v, q_h)}{\|\Pi_h v\|_V}$$
(3.38)

$$= c \sup_{v_h \in V_h / \{0\}} \frac{b(v_h, q_h)}{\|v_h\|_V}$$
(3.39)

The space that will be used to discretize  $H(div, \Omega)$  is the space of the lowest order Raviart-Thomas elements  $(RT_0)$  and for the  $L^2(\Omega)$  the space of piecewise constant functions will be used  $(P_0)$ . These are defined using

$$RT_0(T_i) := \{a + bx : a \in \mathbb{R}^3, b \in \mathbb{R}, x \in T_i\} \subset H(div, \Omega)$$

$$(3.40)$$

$$P_k(T_i) := span\{\Pi_{i=1}^d x_i^{\alpha_i} : x \in T_i, \alpha \in \mathbb{N}^d, \sum \alpha_i \le k\}.$$
(3.41)

where  $T_i$  is a hexahedral or tetrahedral element taken from a valid decomposition T, i.e.  $T_i \in T$ . The entire space can then be defined using

$$RT_0(T) := \{ q \in L^2(\Omega)^3 : q | _{T_i} \in RT_0(T_i), \text{ for all } T_i \in T \}$$
(3.42)

$$P_0(T) := \{ v \in L^2(\Omega) : v | _{T_i} \in P_0(T_i), \text{ for all } T_i \in T \}.$$
(3.43)

For a visualization of the merging finite element check figure (3.1). While the use of  $RT_0$  as a choice for the flux is intuitive, the choice of  $P_0$  for the potential space is not trivial, but one can validate that choice using the following theorem from Vorwerk (2016, p.143).

**Theorem 3** For any tetrahedral/hexahedral element  $T_i$  we have for  $q \in RT_k(T_i)$  that

$$\nabla \cdot q \in P_k(T_i)/Q_k(T_i),$$
  
$$\langle q, n_{\partial T_i} \rangle \in R_k(\partial T_i),$$

with  $R_k(\partial T_i) := \{p \in L^2(\partial T_i) : p|_{f_j} \in P_k/Q_k(f_j), \text{ for all } f_j \in F(T)\}.$ Proof.  $q \in RT_k(T_i)$  can be written as  $q_0 + xp_k$  using  $q_0 \in (P_k/Q_k)^d, p_k \in P_k/Q_k$ , which makes it clear that also  $\nabla \cdot q \in P_k/Q_k$ .

Multiplying q with the face normal vector  $n_{\partial T_i}$ ,  $\langle q, n_{\partial T_i} \rangle = \langle q_0, n_{\partial T_i} \rangle + \langle x, n_{\partial T_i} \rangle p_k$ . Since  $\langle x, n_{\partial T_i} \rangle$  is constant on a face  $f_i$ ,  $\langle q, n_{\partial T_i} \rangle \in P_k/Q_k$ .



Figure 3.1:  $RT_0 \times P_0$  hexahedral elements for 2D (left) and 3d (right) The red point shows the degree of freedom for the  $P_0$ -Space and the blue vectors show the degree of freedom for the  $RT_0$ -Space

That these spaces fulfill Fortin's criterion can be shown due to the ellipticity following from  $RT_0(T(\Omega)) \subset H(div, \Omega)$  and deriving the projection  $\Pi_h$  using (3.16-17) (Braess, 2007, p.143f.).

By using the sets of  $P_0$  and  $RT_0$  basisfunction  $S_h^{P_0}$  and  $S_h^{RT_0}$  respectively, the system (3.16-17) can be written as a system of matrices. For this the discrete solutions  $u_h = \sum_i u_i v_i$ and  $j_h = \sum_k j_k w_k$  are used instead of their analytical counterparts. Here  $v_i \in S_h^{P_0}$  and  $w_k \in S_h^{RT_0}$  were used. Using all this (3.16-17) can be written as:

$$\begin{pmatrix} M & D^T \\ D & 0 \end{pmatrix} \begin{pmatrix} j \\ u \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$
(3.44)

where

$$M_{ij} = \int_{\Omega} \langle \sigma^{-1} w_k, w_j \rangle dx \tag{3.45}$$

$$D_{kj} = \int_{\Omega} v_k (\nabla \cdot w_j) dx \tag{3.46}$$

for  $w_i, w_j \in S_h^{RT_0}$  and  $v_k \in S_h^{P0}$ . What remains is to derive a formula for the source discretization. Here an advantage of the mixed-approach is obvious. Due to  $RT_0$  being used and the source still originating in the flux-space one finds an intuitive formulation using that the right-hand side can be understood as a sum of multiple point dipoles of the form  $j^p = M\delta_{x_0}$ . Here  $M \in \mathbb{R}^3$  denotes the moment of the point dipole and  $\delta_{x_0}$  the Dirac distribution in  $x_0 \in \Omega$ . It should be stated that this approach works analogous to the Whitney-approach but emerges naturally. The source is a singularity which is usually difficult to discretize. (compare Johannes Vorwerk, 2016, chapter 2.10).

This finally leads to

$$b_i = \int_{\Omega} \langle \sigma^{-1}(x) M \delta_{x_0}(x), w_i(x) \rangle dx$$
(3.47)

which could be evaluated by approximating the integral as point evaluation. This would lead to

$$b_i = \langle \sigma^{-1}(x_0) M, w_i(x_0) \rangle \tag{3.48}$$

which could be understood as blurring the singular primary current  $j^p$ . The approach for calculating the right-hand side will be called direct approach. While there is an alternative approach, called the projection approach, the main focus of this thesis will lay on the direct approach. The alternative will still be derived in the following, to ensure a better understanding of the general source modeling, which - in various other cases - is not as intuitive as in the case of the mixed-FEM.

#### 3.5 Projection Approach

The idea of the projection approach is, as the name suggests, based on the concept of projecting the vector-valued current source on to the potential space  $P_0$  using a projection matrix D. Our Matrix D is sufficient by construction as the divergence is in fact a projection onto the potential space. Taking the current source b we can therefore define  $h^{proj} := Db$ .



Figure 3.2: Support of  $h^{proj} = Db^{direct}$  (left) and  $h^{direct} = DM^{-1}b^{direct}$  (right), which has support along the entire domain. src: Vorwerk (2016).

Inserting this definition into (3.44) we then find the problem for the projection approach that has to be solved.

$$\begin{pmatrix} M & D^T \\ D & 0 \end{pmatrix} \begin{pmatrix} j \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ h^{proj} \end{pmatrix}$$
(3.49)

While the theoretical analysis for the projection approach will be done with in this thesis, no implemention and therefore no numerical testing will be done.

#### 3.6 Source-Model for DG and ContG-FEM

Analogous to above the source-models for DG- and ContG-FEM need to be stated and briefly explained. This chapter will not go into detail about the actual construction and will not derive the actual models mathematically. It is merely supposed to give a rough idea on the different models to ensure comparability for later use. The short derivations will be done for the ContG-FEM. More detailed information and the mathematical derivation can be found in Johannes Vorwerk Dissertation (2016), which serves as the basis of this summary.

Once again we find different source-models for the dipole. For the ContG-FEM the partial integration- (PI), the Venant- and the subtraction source models will be used, while for the DG-FEM approach only PI and subtraction-approaches will be used.

#### 3.6.1 Partial Integration

The partial integration source model is based on the intuitive way to solve the singularity problem at hand. Assuming we are using a piecewise-linear testfunction  $h_i$  we can write

$$b_i = \int_{\Omega} (\nabla \cdot j^p) h_i dx. \tag{3.50}$$

Now, similar to Gauss' theorem - and using the distributional derivative of  $\delta_{x_0}$  - one can move the derivative to said testfunction to bypass the singularity of  $j^p$ , i.e.

$$\int_{\Omega} (\nabla \cdot j^p) h_i dx = \int_{\Omega} (j^p, \nabla h_i) dx = \begin{cases} \langle p, \nabla h_i(x_o) \rangle, & if x_o \in supph_i, \\ 0, & else \end{cases}$$
(3.51)

Due to the piecewise-linearity of  $h_i$  the function is almost everywhere differentiable. It should be noted at this point, that due to the testfunction only having support on one meshelement, this discretization could be understood as replacing the dipole by a distribution of monopoles.

#### 3.6.2 Saint Venant

Replacing the dipole by a distribution of monopoles proofs to be a common theme for various different source models. One other of these source models is named after the Saint Venant (see Vorwerk, et al. 2019). The idea used for this approach is, that under certain restrictions, complicated stress distributions can be locally simplified, due to the lack of influence of further distanced contributors. In the case of the EEG- and MEG-problems this means that the point dipole is replaced by a distribution of local monopoles with localized extent  $\rho(x)$ , as long as the moments of the original dipoles are conserved.

The source distribution moments are defined as  ${}^{k}T = \int_{\Omega} (x-x_0)^{k} \rho(x) dx$ . As it is known from physics, this definition does not correspond to the coefficients of the multipole expansion,

which are derived from a multi-dimensional Taylor expansion of u (Jackson et al., 2006). In Cartesian coordinates with  $r = ||x||_2$  and  $x_i$  as the entries of x, q as the sum of charges (which means q = 0 in our case) is reads:

$$u(x) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{\langle p, x \rangle}{r^3} + \frac{1}{2} \sum_{ij} Q_{ij} \frac{x_i x_j}{r^5} + \dots\right)$$
(3.52)

 $Q_{ij}$  and p are the quadrupole and dipole moments respectively, i.e.

$$p = \int x' \rho(x) dx' \tag{3.53}$$

$$Q_{ij} = \int (3x'_i x'_j - r'^2 \delta_{ij}) \rho(x') dx'$$
(3.54)

While it is obvious that, after a transformation to a common origin,  ${}^{1}T$  matches the dipole moment, this is not the case for  ${}^{2}T$  and the quadrupole moments. It can be shown that the differces are negligible (Hanrath et al.). Due to the historical creation of the Venant approach the choice remains as it is given. In order to simplify the calculation of the moments  ${}^{k}T$  once again point-like monopole sources are assumed, i.e.  $\rho = \sum_{i=1}^{n} q_{i} \delta_{x_{i}}$ . The moments can then be rewritten as

$${}^{k}T = \sum_{i=1}^{n} (x_{i} - x_{0})^{k} q_{i} = \sum_{i=1}^{n} \Delta x_{i}^{k} q_{i}$$
(3.55)

where  $\nabla x_i$  denotes the distance between a vertex  $x_i$  and  $x_0$  and  $\nabla \overline{x_i} := \nabla x_i/a^{ref}$  a rescaled version to improve the stability. (3.55) can then rewritten as a matrix system as follows, using the same rescalling convention:

$$\overline{X}_{j} = \begin{pmatrix} (\Delta \overline{x}_{1})_{j}^{0} & (\Delta \overline{x}_{2})_{j}^{0} & \dots & (\Delta \overline{x}_{n})_{j}^{0} \\ \dots & \dots & \dots & \dots \\ (\Delta \overline{x}_{1})_{j}^{k} & (\Delta \overline{x}_{2})_{j}^{k} & \dots & (\Delta \overline{x}_{n})_{j}^{k} \end{pmatrix}$$
(3.56)

$$q = \begin{pmatrix} q_1 \\ \dots \\ q_n \end{pmatrix} \tag{3.57}$$

$$\bar{t}_j = \begin{pmatrix} (^0 \overline{T})_j \\ \dots \\ (^n \overline{T})_j \end{pmatrix}$$
(3.58)

To calculate the load vector q the matrix  $(\overline{W}_j)_{m,s} := (\Delta \overline{x}_m)^r \delta_{m,s}$  for r = 0 or r = 1 is introduced, allowing us to understand q as the minimization of the functional

$$F_{\Lambda}(q) = \|\overline{t}_j - \overline{X}_j q\| + \lambda \|\overline{W}_j q\|_2^2, \qquad (3.59)$$

where the first term penalizes differences between the original source and its approximation, while the second ensures the uniqueness of the solution by additionally penalizing to many large values  $|q_i|$ . Differentiating after  $q_i$  then results in the solution

$$(\overline{X}_j^T \overline{X}_j + \lambda \overline{W}_j^T \overline{W}_j)q = \overline{X}_j^T \overline{t}_j.$$
(3.60)

This then leads to

$$q = \left[\sum_{j=1}^{3} (\overline{X}_{j}^{T} \overline{X}_{j} + \lambda \overline{W}_{j}^{T} \overline{W}_{j})\right]^{-1} \cdot \sum_{j=1}^{3} \overline{X}_{j}^{T} \overline{t}_{j}, \qquad (3.61)$$

which - considering the reference function Z - ends in the Venant RHS:

$$b_i = \begin{cases} q_{Z(i)}, & \text{if } i \in Z^{-1}(0, \dots, n), \\ 0, & else. \end{cases}$$
(3.62)

Simplified this means that this source model blurs the formerly point-like source current.

#### 3.6.3 Subtraction

Reviewing the source models used until now, one could find that all approaches try to substitute the dipole using a distribution of monopoles. The subtraction source-model approaches this in a different manner: Instead of substituting the dipole, the assumption for the existence of a non-empty, open neighborhood  $\Omega^{\infty}$  with constant, isotropic conductivity  $\sigma^{\infty}$  is made, which allows us to remodel the source singularity with an analytical solution into an equation for a correction potential  $u^c$  akin to (3.6).

It can therefore be said that the electric potential u is split into the correction - and the constant, isotropic term:

$$u = u^c + u^{\infty}, \tag{3.63}$$

$$\sigma = \sigma^c + \sigma^{\infty}. \tag{3.64}$$

Using the properties for  $\sigma^{\infty}$  on  $\Omega^{\infty}$ ,  $u^{\infty}$  can be calculated analytically using

$$\sigma^{\infty} u^{\infty} = -(\nabla \cdot j^p) * G. \tag{3.65}$$

Now abusing that

$$\sigma^{\infty} \Delta u^{\infty} = -\Delta((\nabla \cdot j^p) * G) = \nabla \cdot j^p \tag{3.66}$$

and recalling that  $\nabla \cdot j^p = \langle p, \delta_{x_0} \rangle$  a new PDE for  $u^c$  can be stated:

$$\nabla \cdot (\sigma \nabla u^c) = f \text{ in } \Omega \qquad \qquad f := -\nabla \cdot (\sigma^c \nabla u^\infty), \qquad (3.67)$$

$$\sigma \partial_n u^c = g \text{ on } \Gamma \qquad \qquad g := -\sigma \partial_n u^\infty. \tag{3.68}$$

The weak formulation can then be given by: Find  $u^c \in H^1_*(\Omega)$ , such that:

$$\int_{\Omega} \langle \sigma \nabla u^c, \nabla v \rangle dx = -\int_{\Omega} \nabla \cdot (\sigma^c \nabla u^\infty) v dx - \int_{\partial \Omega} (\sigma \partial_n u^\infty) v dx$$
(3.69)

for all  $v \in H^1_*(\Omega)$ . Finally the RHS for the subtraction source-model can be defined.

$$l^{c}(v) := -\int_{\Omega} \nabla \cdot (\sigma^{c} \nabla u^{\infty}) v dx - \int_{\partial \Omega} (\sigma \partial_{n} u^{\infty}) v dx \qquad (3.70)$$

Regardless of the source model used, a system of equations has to be solved for the calculated matrix system. Due to the large size of the system numerical solvers need to be used. Therefore the following chapter will introduce the conjugate-gradient (CG) solver and various different preconditioners to optimize its use for the problem at hand.

#### CHAPTER 4

## Solvers

#### 4.1 Introduction

In general it is inefficient to directly invert the system matrix A. While A itself is a sparse matrix, the inverse matrix  $A^{-1}$  would end up being a dense matrix. Aside of the memory-problems for very short mesh-width h the time requirement for inverting the matrix A is insufficient. Due to these problems, systems Ax = b are solved using different methods. This chapter will deal with only one approach: the conjugate gradient method (CG-Method). Using only the unpreconditioned CG-solver will not be able to solve our system efficiently, due to the structure of our matrix. Not only is the condition of our system far from optimal, also the problem we are trying to solve is a saddle-point problem, for which Krylow-subspace solvers own a slow convergence-rate.

Even using common preconditioners directly on our system often fail due to the structure of our equation. This problem can be avoided by calculating the schur-complement of our original system. After explaining the fundamentals necessary, the schur-complement solver will be explained which will be the implemented solver used for calculating the solution of our problem.

```
#include <iostream>
1
2
3
    int main()
4
    {
\mathbf{5}
      vector x;
6
      x = 0;
7
      vector Ax0 = A.matmul(x);
8
      vector r = b - Ax;
      double res = norm(r);
9
10
      vector d = r;
11
      for (int k = 0, k<maxiter || tol>res, ++k)
12
      ſ
13
        // save product once
14
        vector z = Ad;
        int rTr = r.transpose() * r;
15
16
17
        int alpha = rTr / (d.transpose()*z)
18
        x += alpha*d;
        r -= alpha*z;
19
20
21
        int beta = r.transpose()*r / rTr;
22
        d = r + beta*d;
23
        res = norm(r);
24
      }
25
   }
```

Algorithm for the unpreconditioned CG-Method

#### 4.2 Conjugate Gradient

The conjugate gradient method (CG-Method) is an iterative method for solving large linear equations Ax = b which are symmetric and positive definite. As known, a solution of Ax = bis at the same time a unique minimizer of the quadratic function  $E(x) = \frac{1}{2} \langle Ax, x \rangle - \langle b, x \rangle$ . The gradient of this functional is given by  $\nabla E|_x = Ax - b = -r$ , i.e. the negative residual. This gradient can therefore be calculated quickly for sparse matrices. In contrast to other gradient-descend methods however, the CG-method reduces not in the direction of the residuum but in the directions of A-conjugate vectors  $d_k$ . This means  $\langle Ad_i, d_j \rangle = 0$  for  $i \neq j$ . For a starting vector  $x_0$  one can then construct the Krylow-space between A and  $r_0$ as

$$K_k := x_0 + span\{r_0, Ar_0, \dots, A^{k-1}r_0\}.$$
(4.1)

The necessary space can therefore be calculated automatically in the algorithm. Finally it can be shown that, since  $K_k 0$  is the span of A-conjugate vectors, that  $dim(K_k) = k$ . This also leads to the fact, that for a matrix  $A \in \mathbb{R}^{n \times n}$ , the CG-method converges after niterations (assuming exact calculations for each iteration). However for big matrices it is usually unrealistic to run n iterations, therefor the actual convergence rate is interesting.

**Theorem 4** The convergence rate for the CG-method in step k with solution  $x_k$  is emerging and the exact solution x is given by

$$\|x_k - x\|_A = 2\left(\frac{\sqrt{\kappa}(A) - 1}{\sqrt{\kappa}(A) + 1}\right)^k \|x_0 - x\|_A$$
(4.2)

Proof Hackbusch, W. (1994)

Therefore the iterations necessary are directly correlated to the condition of the matrix. Improving the condition is therefore the most efficient way in reducing the time needed to solve the system to a satisfying degree.

#### 4.3 Preconditioners

Preconditioners are a fundamental necessity for solving bad-conditioned problems in a realistic setup. While the CG-method itself converges as long as the matrix is either symmetric and positive definite or by solving the problem  $A^tAx = A^tb$  instead, the time required to do so will not be sustainable in any realistic scenario. As shown above, in order to improve the rate of convergence, one needs to improve the condition of the original system Ax = b. This can be done by finding an alternative system  $\tilde{A}x = \tilde{b}$  which approximates our original problem. The right approximation depends on the original problem at hand. In this case three different preconditioners will be presented: An incomplete lower-upper (ILU0) -, an algebraic multi-grid (AMG) - and a Mass-Lumping preconditioning, which will all be used for our problem.

#### 4.3.1 AMG-Preconditioner

The algebraic-multi-grid (AMG) preconditioner is an alternative approach to usual multigrid(MG) preconditioner, which is based on the idea to smooth the residuum by applying few iterations of a solver which eliminates the portion of the error, corresponding to high frequencies, leaving only the error on the lower frequencies, which can be approximated, by prolonging the error of a lower resolution grid. This means that afterwards the system is restricted to a grid with larger stepwitdh h, which allows the approximation of a smoothed residuum, which can then be prolonged to the original grid (see Wolters, et al. 2002 and Lew, et al. 2009). This can be done by multiple stages, creating a hierarchy of grids, resulting in low computational costs at the lowest level. Due the initial smoothing the remaining residuum on the highest level is itself smooth and can therefore be easily corrected with the prolonged residuum of the lower level. The remaining residuum is therefore free of low and high frequency errors. The advantage of this approach is, that a much higher convergence rate can be reached for systems with low portions of higher frequency errors.

Due to the difficulty of this approach being to create the grid hierarchy for complex geometries of the domain  $\Omega$  and due to difficulties tuning the high-frequency smoothers and optimizing the restriction-operator for multigrid-CG, the goal of the AMG-preconditioner is to avoid said problems. Instead of creating a hierachy of grids, which approximate the original geometry the system, a subset of the unknowns is chosen and the matrix is restricted using a algebraic chosen restriction, which is only based on the information taken from the matrix.

However the choice of the restriction (which is itself the transpose of the prolongation) is not trivial.

```
1
    #include <iostream>
 \mathbf{2}
3
    int main()
 4
    {
\mathbf{5}
      vector x;
6
      x = 0;
      vector Ax0 = A.matmul(x);
 7
 8
      vector r = b - Ax;
      vector h = Ch;
9
10
      double res = norm(r);
      vector d = r;
11
      for (int k = 0, k<maxiter || tol>res, ++k)
12
13
      Ł
14
         // save product once
15
         vector z = Ad;
        int rTh = r.transpose() * h;
16
17
        int alpha = rTh / (d.transpose()*z)
18
        x += alpha*d;
19
20
        r -= alpha*z;
21
        h = Cr:
22
23
        int beta = r.transpose()*h / rTh;
24
        d = h + beta*d;
25
        res = norm(r);
26
      }
27
    }
```

Algorithm for the preconditioned CG-Method

#### 4.3.2 SSOR-Preconditioner

One of the two preconditioners used for solving the given problems is the symmetric successive over-relaxation (SSOR) which is special case of the successive over-relaxation (SOR), which is in itself a variant of the Gauss-Seidel Method

The SOR-method itself can be described using the algorithm stated below. However, since the preconditioner is created using a symmetric matrix – which one needs for the CG-method – the symmetric version emerges that can be described using

$$P = \left(\frac{D}{\omega} + L\right) \frac{\omega}{2 - \omega} D^{-1} \left(\frac{D}{\omega} + U\right), \tag{4.3}$$

where A = (L + D + U), L, D and R denote the lower, diagonal and upper submatrix respectively. This then leads to the iterative

$$x^{k+1} = x^k - \gamma^k P^{-1} (Ax^k - b), \text{ for } k \ge 0.$$
(4.4)

This is however not the usual application of the matrix, since inverting P would not be efficient once again. Applying the preconditioner to both sides of the equation or onto the residuum ensures the raise of the original solution when applying the CG-method onto the altered system.

#### 4.4 Schur-complement Solver

In general the schur-complement is calculated for a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

If A or C are invertible we can multiply the row  $A^{-1}$  or  $C^{-1}$  respectively and subtract it from the remaining row in order to gain the Schur-complement S. In the case of this thesis, we can understand equation (3.44) as two equations

$$Mj + D^T u = h_1 \tag{4.5}$$

$$Dj = h_2 \tag{4.6}$$

where  $(h_1, h_2)$  equals either  $h_{proj}$  or  $h_{direct}$ . If we now eliminate D in (4.6) by multiplying  $DM^{-1}$  with the first the equation on to it, we gain

$$Su := -DM^{-1}D^T u = h_2 - DM^{-1}h_1 (4.7)$$

where  $S = -DM^{-1}D^T$  denotes the Schur-complement. Once the electric potential u was calculated using the equation above, the current j can be calculated by solving

$$Mj = h_1 - D^T u. (4.8)$$

In order to solve (4.7) one will once again use the CG-Solver. Therefore one needs to assure, that S is a symmetric, positive definite matrix. This however was done by Braess (2007).

**Theorem 5** S is positive definite. If M is symmetric, also S is symmetric. Proof. Proof.

Braess (2007, p. 216)

For solving system (4.7) it is necessary to solve  $M^{-1}D^T u = y$ , which can be multiplied with M leading to  $My = D^T u$ , which can be solved using the CG-Method as described above. In total, we therefore need to solve three different systems:

$$My = h_1 \tag{4.9}$$

$$Su = h_2 - Dy \tag{4.10}$$

$$Mj = h_1 - Du. (4.11)$$

At this point what is left it to state which preconditioner will be used at which place. While M is a sparse matrix, which makes it easy to use the SOR method, heavily improving the convergence rate, S does not have to be sparse. Therefore for this matrix the AMG-method will be used. Due to the fact that the structure only needs to be calculated once for multiple dipoles, which will always be used in realistic or relevant setups, the computational cost is acceptable.

#### **CHAPTER 5**

## Implementation

To transfer the mathematical idea into actual code structures to realize said ideas, preexisting structures are necessary. In order to recreate the results of this thesis the setup has therefore to be transparent. For the recreation four major topics are relevant:

Which software setup is used as basis for the Mixed-FEM code, how are the matrices and the right-hand side assembled in said setup, which additional optimizations are used and which tests are used to create the different results.

#### 5.1 Software

The implementations which are done for this thesis are based on the preexisting duneurotoolbox (http://duneuro.org/), which itself is based on the dune-toolbox (https://www.dune-project.org/).

Duneuro can be used to calculate solutions for the EEG- and MEG-forward problems using continuous and discontinuous finite element methods (CG and DG) for various different source models, like the subtraction, Venant or Whitney approaches. Those approaches will be used later in this work as reference points for comparing the results of the mixedapproach. The preexisting basis functions for the reference element, required for assembling the system matrix, originate in the Dune-PDELab module.

#### 5.2 Assembling the system matrix

Various problems can emerge when trying to assemble the system matrix. The first problem at hand is finding the basis-functions for the RT0-space. While the functions for the P0space emerge trivially, the basis for the former would have to be calculated using an equation system. This however can be avoided by switching to local basis functions defined on the reference element  $E_i$ . To do this, two major steps are necessary: the application of the Piola transformation to transform the basisfunctions to their reference element counterpart and transforming the integral itself. The transformation onto the reference element is shown in figure (5.1) using the functional  $F_i$ , corresponding to an finite element  $T_i$ . The functional  $F_i$  which project a tetraedric element  $T_i$  with corners  $x_1$ ,  $x_2$  and  $x_3$  for example could be calculated using

$$F_i^{\Delta}(\psi) := x_1 + \left(x_2 - x_1 | x_3 - x_1\right)(\psi)$$
(5.1)

For greater dimensions or different finite element types different equations emerge, which



Figure 5.1: Illustration of functions for transferring the respective refence-elements onto specific elements taken from the grid.

however can be automated by deriving the necessary equation, which only depends on the finite element type.

If we now assume a transformation onto the reference Element  $E_{ref}, F_i$  for each element  $T_i \in T$ , where T denotes the valid decomposition of  $\Omega$  into finite Elements  $T_i$ , we can, using the structure of M from (2.45), transform said integral:

$$M_{kj} = \int_{\Omega} \langle \sigma^{-1} w_k, w_j \rangle dx \tag{5.2}$$

$$=\sum_{i=1}^{n}\int_{T_{i}}\langle\sigma^{-1}w_{k},w_{j}\rangle dx$$
(5.3)

$$= \sum_{i=1}^{n} \int_{T_i} \langle \frac{1}{|J_i|} \sigma^{-1} J_i \overline{w}_k, \frac{1}{|J_i|} J_i \overline{w}_j \rangle dx$$
(5.4)

$$= \sum_{i=1}^{n} \frac{1}{|J_i| |J_i|} \int_{T_i} \langle \sigma^{-1} J_i \overline{w}_k, J_i \overline{w}_j \rangle dx$$
(5.5)

$$= \sum_{i=1}^{n} \frac{1}{|J_i| |J_i|} \int_{E_{ref}} \langle \sigma^{-1} \overline{w}_k, G_t \overline{w}_j \rangle |J_i| dx$$
(5.6)

$$= \sum_{i=1}^{n} \frac{1}{|J_i|} \int_{E_{ref}} \langle \sigma^{-1} \overline{w}_k, G_t \overline{w}_j \rangle dx$$
(5.7)

with

$$J_i^{\triangle} := det((x_2 - x_1 | x_3 - x_1))$$
(5.8)

where  $G_t = ((t_i \cdot t_j))$  is the covariant metric tensor with  $||G_t|| = 1$ .  $t_i$  denote the vector-basis for constructing the normal vectors. Here  $\overline{w}$  denotes the basis functions on the reference element. We can then continue for the discrete divergence:

$$D_{kj} = \int_{\Omega} v_k (\nabla \cdot w_j) dx \tag{5.9}$$

$$= \sum_{i=1}^{n} \int_{T_i} v_k (\nabla \cdot w_j) dx \tag{5.10}$$

$$= \sum_{i=1}^{n} \int_{T_i} v_k \left(\frac{1}{|J_i|} \overline{\nabla} \cdot \overline{w}_j\right) dx \tag{5.11}$$

$$= \int_{T_k} \frac{1}{|J_k|} (\overline{\nabla} \cdot \overline{w}_j) dx \tag{5.12}$$

$$= \int_{E_{ref}} \frac{1}{|J_k|} (\overline{\nabla} \cdot \overline{w}_j) |J_k| dx \tag{5.13}$$

$$= \int_{E_{ref}} (\overline{\nabla} \cdot \overline{w}_j) dx \tag{5.14}$$

Now that the left-hand sides can be assembled using only the local basis functions, the right-hand side is the only part missing. For this we need to transform not only the basis function but also the Dirac distribution  $\delta_{x_0}$ , resulting in an additional  $1/|J_i|$ .

$$b_j = \int_{\Omega} \langle \sigma^{-1} m \delta_{x_0}, w_j \rangle dx \tag{5.15}$$

$$= \sum_{i=1}^{n} \int_{T_i} \langle \sigma^{-1} m \delta_{x_0}, w_j \rangle dx \tag{5.16}$$

$$= \sum_{i=1}^{n} \int_{T_i} \frac{1}{|J_i|} \langle \sigma^{-1} m \delta_{x_0}, J \overline{w}_j \rangle dx$$
(5.17)

$$= \sum_{i=1}^{n} \int_{E_{ref}} \frac{1}{|J_i|^2} \langle \sigma^{-1} m \delta_{\tilde{x}_0}, J \overline{w}_j | J_i | \rangle dx$$
(5.18)

$$= \sum_{i=1}^{n} \int_{E_{ref}} \frac{1}{|J_i|} \langle \sigma^{-1} m \delta_{\tilde{x}_0}, J \overline{w}_j \rangle dx$$
(5.19)

To calculate the various integrals fitting quadrature formulas are necessary. The integral for b do not have to be calculated, since it will be approximated using a point-evaluation. As the term we interpolate for M is a composition of two linear functions the required integration order is two. Since one can restrict the domain over which is integrated to the specific element  $T_i$  when calculating  $D_{ij}$  the P0-basis function gets obsolete, which enables a integration of order one, since only the divergence of a linear function remains.

While the matrices can now be directly assembled, the overall performance of the code still can be optimized, by calculating so called transfer matrices for the respective problems.

#### 5.3 Transfer matrices

Computational efficiency is a driving factor when applying mathematical methods to realistic scenarios. In order to reach this efficiency we want to apply the transfer matrix approach which revolves around the idea, that - regardless of the actually numerical approach - the structure of the problem one needs to solve to calculate the electric potential u always looks like Au = b. For a hexaedric grid with a mesh-size of 1mm the dimension of A usually ranges at around 3 million unknowns for a realistic head model. For the most common inverse approaches a source space has to be chosen. This discrete set of dipole positions and directions is used to approximate the sources of the neural activity. A leadfield matrix L is therefore computed which consists of the forward solution for a pair of a source space position  $p_i$  and direction  $m_i$  as respective column. The number of columns is therefore given by the number of positions multiplied by the number of directions (m \* p). Due to the great dimension of the systemmatrix A and up to thousands of possible dipole positions the computational cost necessary to calculate all solutions for assembling the leadfield matrix is very expensive.

The number of sensors S usually ranges within the hundreds and is therefore a lot lower then the number of possible dipole positions. This fact motivated the transfer matrix approach. As we are only interested in the unknown values of u at the sensor positions one can precalculate a restriction matrix R which can either be based on a point evaluation leading to a reduced identity matrix, or be based on an approximation of the surface area of the respective sensor. Using all this one can then proceed for the EEG and MEG problems as follows to calculate the complete transfer matrix, which enables us to evaluate the electic potential u very efficiently.

#### 5.3.1 EEG

Let I denote the set of indices for an enumeration of the sensors. For each  $i \in I$  we want to evaluate our electrical potential u using a functional  $E_i$ .

$$q_i = E_i u$$
$$\Leftrightarrow Q = E u$$

Instead of doing this calculation using the electric potential we want to multiply with the original right-hand side h. For the different discretizations  $h_{proj}$  and  $h_{direct}$  different solutions emerge. Using that  $h_{proj} = Db$  is the product of the discrete divergence and our source, that  $h_{direct} = DM^{-1}b$  and that  $S = DM^{-1}D^T$  denotes our Schur-complement system we can rewrite the equation as:

$$Q = ES^{-1}h =: Th. (5.20)$$

Solving for T:

$$T = ES^{-1}$$
  
$$\Leftrightarrow T^{T} = S^{-T}E^{T}$$
  
$$\Leftrightarrow S^{T}T^{T} = E^{T}.$$

Proceeding further leads to the formulations:

$$T_{proj}b := TDb \tag{5.21}$$

$$T_{direct}b := TDM^{-1}. (5.22)$$

$$T_{direct} = TDM^{-1}$$
  

$$\Leftrightarrow T_{direct}M = TD$$
  

$$\Leftrightarrow M^TT_{direct} = D^TT^T$$
  

$$\Leftrightarrow MT_{direct} = D^TT^T$$

The problem can be avoided by substituting  $M^{-1}$  with  $C^{-1}$  which was used for the CGsolver of the outer iteration and had to be calculated anyway. The advantage is that  $DC^{-1}b$ is closer to  $h_{direct}$  and therefore supports the grid structure better.

#### 5.3.2 MEG

We remember that the solution of the Poisson equation (2.23)

$$A(r) = \frac{\mu_0}{4\pi} \int_{\Omega} \frac{j(r')}{|r-r'|} d^3 r'$$
(5.23)

is leading to

$$B(r) = \frac{\mu_0}{4\pi} \int_{\Omega} j(r') \times \frac{r - r'}{|r - r'|^3} d^3 r'.$$
(5.24)

We therefore use an evaluation of the functional  $\Sigma_i$  using the current j. Due to the primary magnetic flux  $\Phi_P$  being calculated analytically, it is sufficient to calculate the secondary magnetic flux  $\Phi_S$ . For this calculation we can use either  $j_{direct}$  or  $j_{proj}$ , depending on our h used in the introduction or our source model.

$$j_{direct} = M^{-1}(b - D^T u) (5.25)$$

$$j_{proj} = -M^{-1}D^T u. (5.26)$$

Starting for  $j_{proj}$  we can then proceed as above in order to calculate the transfer matrix  $T_{proj}^{meg}$  as follows:

$$\Phi_s = \Sigma j_{proj}$$
  
=  $-\Sigma M^{-1} D^T u$   
=  $-\Sigma M^{-1} D^T S^{-1} h_{proj}$   
=  $-\Sigma M^{-1} D^T S^{-1} D b$   
=  $T_{proj}^{meg} b.$ 

As we want to calculate  $T_{proj}^{meg}$  we therefore proceed similar to the case of the EEG-transfer matrix. In the following the MEG-Transfer matrix will be noted without the *meg* notation to keep it simple and will be based on applying  $h_{proj}$  instead of b which will be the actual application.

$$T = -\Sigma M^{-1} D^T S^{-1}$$
  
$$\Leftrightarrow T^T = -S^{-T} D M^{-T} \Sigma^T$$
  
$$\Leftrightarrow T^T = -S^{-1} D M^{-1} \Sigma^T$$
  
$$\Leftrightarrow ST^T = -D M^{-1} \Sigma^T$$

At this point we first have to calculate the right-hand side for which we require to solve  $Y = M^{-1}\Sigma^T$ , which we can write as  $MY = \Sigma^T$ . Once we calculated Y and therefore were able to calculate T one can directly deduct our actual transfer matrix  $T_{proj}^{meg}$ .

$$\Phi_s = Th_{proj}$$

$$= TDb$$

$$= T_{proj}^{meg}b$$
(5.27)

Let us now take a closer look at the case of  $j_{direct}$ .

$$\Phi_s = \Sigma j_{direct}$$
  
=  $\Sigma M^{-1}(b - D^T u)$   
=  $\Sigma M^{-1}(b - D^T S^{-1} h_{direct})$   
=  $\Sigma M^{-1}(b - D^T S^{-1} D M^{-1} b)$   
=  $\Sigma M^{-1}(Id - D^T S^{-1} D M^{-1})b$ 

We need to therefore introduce two support matrices  $T_1$  and  $T_2$ .

$$T_1 := \Sigma M^{-1} T_2 := -\Sigma M^{-1} D^T S^{-1} D M 1 - 1$$

While  $T_1$  can be calculated by solving the system  $MT_1^T = \Sigma$ ,  $T_2$  needs some further explanation.

$$T_{2} = -\Sigma M^{-1} D^{T} S^{-1} D M 1 - 1$$
$$T_{2} = T_{proj} M^{-1}$$
$$T_{2} = M^{-1} T_{proj}^{T}$$
$$\Leftrightarrow M T_{2}^{T} = T_{proj}^{T}$$

Using the result from above we can conclude that we can solve for  $T_2^T$  using  $T_{proj}^T$ . Finally it can be seen that  $T_{proj} = T_1 + T_2$ .

#### 5.4 Experiments

What remains to do is to test how the two different mixed-FEM approaches perform, compared to CG-FEM and DG-FEM, in their relative errors, their magnitude errors and their relative difference measure (RDM).

For this the solutions of ContG- and DG-FEM for different source model discretizations will be compared to those of the mixed-approach using the direct source model. The ones used for ContG- and DG-FEM will be the common variants, namely the venant-, the subtractionand the partial-integration source models, where the venant- source model will only be used for the ContG-FEM approach. Using a spherical domain, consisting of 4 compartments, enables the comparison with the analytical solutions stated in (2.28) and (2.29). The domain



Figure 5.2: Illustration of the test case for a simple grid. Red stands for lower conductivities and dark blue for higher. The brighter blue dot shows a possible location for a dipole.

can be further described using the center of the domain, given at the point (127, 127, 127), the outer radii 78 (brain), 80 (CSF), 86 (skull), 92 (skin )and its conductivites 0.33 S/m, 1,79 S/m, 0.01 S/m, 0.43 S/m.

Following the structure used throughout this thesis the finite elements used to discretize this domain will be hexaedric, without node-shift. Two meshes of different refinement of 1mm or 2mm will be used, where the former approximates the geometry of the grid better. For the calculation the source will consist of 500 to 10000 different dipoles with varying eccentricity. The electric data for the EEG calculation will be based on measurements on up to 70 electrodes, regularly distributed over the whole sphere. The MEG calculations will use a similar amount of coils for the measurement data. The setup therefore looks similar to the following visualizations.

The calculations for the EEG forward problem will be done using the formerly derived and implemented transfer matrix approach, while the MEG problem will be solved calculating the solution for the forward problem for each dipole respectively.

## Evaluation and Conclusion

What lastly remains to be done is that the tests just described are resolved and that the results are evaluated graphically and literally. Finally it will be concluded whether or whether not, the direct source model approach is comparable to the accuracy of the usual approaches. The hardware used for the computations will be briefly described to enable comparability for replications of the test-cases on different devices. Lastly, a quick outlook will be given on which questions and problems arose within the theory of this thesis and its implementations.

#### 6.1 Hardware

As stated in the short chapter introduction the hardware used will be quickly described. All the calculations were done on the same computer-system consisting of an Intel Core i7 105100U CPU octacore 2.3 GHz processor, with 16G RAM. The software was installed and developed on Ubuntu 18.04 64x bit operating system.

#### 6.2 Evaluation

As stated in chapter 5, different source-models were tested. First the relative errors (RE) for different source-models should be compared to give a rough understanding on the actual accuracy of the numerical approaches. When looking at Table 6.1

Table 6.1: Average relative error for different source models for the EEG forward problem, using 10000 dipoles, for greater eccentricities and a mesh size of 2mm.

Source model	Ecc 0.8	Ecc 0.9	Ecc 0.95	Ecc 0.99
ContG PI	0.0973403118	0.103002964	0.1057619916	0.237377436
ContG Subtraction	0.0968255298	0.1026606788	0.1062498244	0.5191629296
ContG Venant	0.0967298584	0.1026234504	0.1065494	0.1864244312
DG PI	0.1005805936	0.1062139472	0.1098324308	0.2461023388
DG Subtraction	0.100078984	0.1059095796	0.1100331418	0.5428161502
Mixed Direct	0.102959608	0.1207144254	0.133478122	0.3682835944

one quickly finds that the accuracy for the ContG- and DG-FEM approaches may be similar for all ranges of eccentricity (aside of the subtraction source models, which obviously are incorrect) but the accuracy of the direct approach for the mixed-FEM approach is only comparable at very high eccentricities, where the error is extremely high anyway and there this fact is negligible.

For lower eccentricities similar results can be found. When looking at table 6.2 one finds that the accuracy of the direct approach is, once again, half as accurate as the other approaches.

Source model	Ecc 0.5	Ecc 0.6		
ContG PI	0.0849444032	0.0880041648		
ContG Subtraction	0.084877648	0.0879272642		
ContG Venant	0.0848537188	0.0879173436		
DG PI	0.0880963846	0.0913670358		
DG Subtraction	0.088025344	0.091296976		
Mixed Direct	0.092764188	0.0949681714		

Table 6.2: Average relative error for different source models for the EEG forward problem, using 10000 dipoles, for smaller eccentricities and a mesh size of 2mm.

The convergence-rate of the model is mathematically equal to that of the ContG- and DG-FEM source models. Due to testing issues with the subtraction source model for both cases, the data will only be compared between the ContG PI and Venant source models. The results (6.3) however match the expected convergence rate. The magnitude is therefore equal to that of the other approaches.

Source model	Ecc 0.8 2mm	Ecc 0.9 2mm	Ecc 0.8 1mm	Ecc 0.9 1mm	
ContG PI	0.0973403118	0.103002964	0.0477950686	0.05076350776	
ContG Venant	0.0967298584	0.102623450	0.0478205012	0.0507333596	
Mixed Direct	0.102959608	0.1207144254	0.06007078627	0.06427468982	

Table 6.3: Comparison of the scenario above with mesh size 1mm.

Finally for the EEG-problem the actual lnMAG- and RDM-errors for different eccentricities (namely 0.6, 0.8, 0.9 and 0.95) should be analyzed. Here the red, blue and green box plots correspond to the mixed-direct-, ContG-PI and the ContG-Venant source models, in case of a 2mm mesh with 10000 dipoles.



Once again it is found that, while the magnitude of the errors of the direct approach is equal that of the other approaches, the actual accuracy is still inferior to the classic approaches.



What remains to be seen is the result of the data for MEG-problem. Since the data of the MEG-problem relies on the solution of the EEG forward problem, similar results would be expected. Due to the failed calculations for the subtraction approach, once again the data

will	only	be	compared	to	the	ContG-FEM	approaches.
	•/		1				11

Table 6.4: Average relative error, using a mesh-size of 2mm and 10000 dipoles.

Source model	Ecc 0.8	Ecc 0.9
ContG PI	0.00984436994	0.01311309488
Mixed Direct	0.0254471461	0.03537295776

Table 6.5: lnMAG error, using a mesh-size of 2mm and 10000 dipoles.

Source model	Ecc 0.8	Ecc 0.9
ContG PI	0,0000794131048	0.000251250934
Mixed Direct	0.00077969542888	0.0002037855774

Table 6.6: RDM error, using a mesh-size of 2mm and 10000 dipoles.

Source model	Ecc 0.8	Ecc 0.9
ContG PI	0.00839845362	0.01111331828
Mixed Direct	0.0232491973	0.03153935264

Once again one finds that, even though the differences are almost neglible in size, the common approach of the ContG-PI outperforms the direct approach in all of the classic error-measures.

#### 6.3 Conclusion

Looking at the results found for the EEG- and MEG-problems the actual performance of the mixed-FEM approach with the direct source model proves to be lacking compared to commonly used approaches. Neither the computational time, nor the actual accuracy legitimate the use of the approach over the common practice. While Johannes Vorwerk showed in this thesis, that the direct approach is capable or reducing potential-leakages when calculating near the skill tissue, the DG-FEM still proves to be more the efficient and accurate approach.

While the computational cost for the EEG-Problem was greatly reduced due to the help

of the transfer matrix approach, the interesting case where the performance-advantages for the approach would come into place, would have been the MEG-problem using the same approach. Calculating the current j together with the electric potential u greatly reduces the additional cost when solving the MEG forward problem. Due to the additional calculations however, the approach needs more storage-space while still taking longer to converge.

In case of the MEG-Problem the direct source model approach was able to stay relative to the use of ContG- or DG-FEM source models, while still not being able to surpass their accuracy. Further testing on the stability of the solution should however be done, due to the different used methods.

#### 6.4 Outlook

What remains to be done is to briefly explain further topics, which could be reviewed and studied. The main problem of this thesis was to recreate the projection approach results from Johannes Vorwerk Dissertation. This was caused due to him originally using the wrong scaling factor for the right-hand side. Additionally the tensors were not correctly used and implemented, leading to additional errors during the calculation. The first thing that could be done in succession to this thesis is to derive the correct mathematical scaling factor for the projection approach and produce new results to compare with the direct approach given above.

Another possible approach is to implement the MEG-transfer matrix approach, which was already theoretically derived. Further testing could be used to evaluate the actual efficiency of this method.

Two, more theoretical, approaches are for example further stability analysis on non-structured grids or the application of different, potentially non-symmetric, solvers.

## Bibliography

Braess, D. (2007). Finite Elemente in der Mechanik elastischer Körper. Finite Elemente: Theorie, schnelle Löser und Anwendungen in der Elastizitätstheorie, 271-340.

De Munck, J. C., and Maria J. Peters. "A fast method to compute the potential in the multisphere model." IEEE Trans. Biomed. Eng 40.11 (1993): 1166-1174.

Fortin, M., & Brezzi, F. (1991). Mixed and hybrid finite element methods. New York: Springer-Verlag.

Hackbusch, Wolfgang. Iterative solution of large sparse systems of equations. Vol. 95. New York: Springer, 1994.

Jackson, J. D. (2006). Klassische Elektrodynamik, 4De Gruyter. Berlin, New York.

Lew, S., et al. "Accuracy and run-time comparison for different potential approaches and iterative solvers in finite element method based EEG source analysis" NIH Public Access (2009)

Piastra, M. C. (2019). New finite element methods for solving the MEG and the combined MEG/ EEG forward problem (Doctoral dissertation, Universitäts-und Landesbibliothek Münster).

Sarvas, Jukka. "Basic mathematical and electromagnetic concepts of the biomagnetic inverse problem." Physics in Medicine & Biology 32.1 (1987): 11.

Vorwerk, J. (2016). New finite element methods to solve the EEG/MEG forward problem (Doctoral dissertation, Westfälische Wilhelms-Universität Münster).

Vorwerk, Johannes, et al. "The multipole approach for EEG forward modeling using the finite element method." NeuroImage 201 (2019): 116039.

Wolters, Carsten H., et al. "A parallel algebraic multigrid solver for finite element method based source localization in the human brain." Computing and visualization in science 5.3 (2002): 165-177.

Hiermit versichere ich, dass die vorliegende Arbeit mit dem Titel Application of the mixed-FEM approach for the MEG-Problem selbstständig verfasst worden ist, dass keine anderen Quellen und Hilfsmittel als die angegebenen benutzt worden sind und dass die Stellen der Arbeit, die anderen Werken – auch elektronischen Medien – dem Wortlaut oder Sinn nach entnommenen wurden, auf jeden Fall unter Angabe der Quelle als Entlehnung kenntlich gemacht worden sind.

Münster, February 9, 2022

(Reiner Stubbemann)

Ich erkläre mich mit einem Abgleich der Arbeit mit anderen Texten zwecks Auffindung von Übereinstimmungen sowie mit einer zu diesem Zweck vorzunehmenden Speicherung der Arbeit in eine Datenbank einverstanden.

Münster, February 9, 2022

(Reiner Stubbemann)