Sparse Recovery Conditions and Realistic Forward Modeling in EEG/MEG Source Reconstruction

Felix Lucka, Sina Tellen, Carsten H. Wolters, Martin Burger

(1) Institute for Applied Mathematics (2) Institute for Biomagnetism and Biosignanalysn (3) Cells in Motion Cluster of Excellence. All: University of Münster, Germany

Background: EEG/MEG Source Reconstruction

Measuring the induced electromagnetic fields at the head surface to estimate the instantaneous, underlying, activity-related ion currents in the brain (instantaneous/static EEG/MEG source reconstruction) is a challenging, high-dimensional, severely ill-posed inverse problem:

\[ (IP) \quad Ax = b, \quad b \in \mathbb{R}^m, \quad x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{m \times n} \]

where \( b \) represents the measured data at 74 EEG or 273 MEG sensors (Figure 1), \( x \) represents the amplitudes of the discretized current field at \( n \) source locations distributed in the gray matter (Figure 4) oriented in normal direction of the cortical surface (normal constraint). A common source density is \( n = 8000 \). Computing the system matrix \( A \) requires constructing a model of the head’s tissues (head model, see Figure 2) and solving the underlying PDEs on it (forward computation).

Figure 1: EEG (left) and MEG (right) sensors.

Figure 4: \( n = 8000 \) discrete sources (black cones) visualized on a corresponding partition inversion window. See Tellen, 2013 for a detailed description of the source space generation.

Motivation

Using spatial sparsity to solve (IP) has become popular in EEG/MEG (e.g., Lucka et al., 2012, Gramfort et al., 2013).

We are especially interested in the interplay of realistic forward and sparse inverse modeling

- Focus on how the intrinsic recovery properties of \( A \) evolve with modeling complexity (not on modeling errors!)
- Examined for \( \ell_1 \)-norm but not for \( \ell_1, \ell_\infty \)-norm or hierarchical Bayesian modeling approaches (Lucka et al., 2012). Dependence on source density \( \gamma \) Spatial resolution of sparse EEG/MEG?
- Main problems: Source separation and localization.
- Suitable framework/tools for our examinations? Concepts from compressed sensing?

Uniform and Nonuniform Recovery Conditions

We want to recover the \( k \)-sparse solution \( x \) (support set \( I \)) of

\[ \begin{align*}
(\text{L0}) \quad & \min_{x \in \mathbb{R}^n} \|x\|_0 \quad \text{s.t.} \quad Ax = b \\
(\text{L1}) \quad & \min_{x \in \mathbb{R}^n} \|x\|_1 \quad \text{s.t.} \quad Ax = b
\end{align*} \]

from the solution of

\[ \begin{align*}
(\text{RIP}) \quad & \left( 1 - \delta_k \right) \|x\|_2^2 \leq \|Ax\|_2^2 \leq \left( 1 + \delta_k \right) \|x\|_2^2 \quad \text{for all } x \in \mathbb{R}^n.
\end{align*} \]

Uniform recovery guarantees the recovery of all \( k \)-sparse \( x \). The strongest relies on the coherence of the matrix \( A \):

\[ (\text{Cho}) \quad \kappa := \frac{\mu_2(A)}{\mu_1(A)} \]

Where \( \mu_i(A) := \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Ax\|_2}{\|x\|_2} \quad (i \geq 1) \quad \mu_0(A) := \|A\|_0 \]

Weakens conditions rely on the restricted isometry property of \( A \), i.e., the smallest number \( \delta_k \) s.t.

\[ (\text{RIP}) \quad \left( 1 - \delta_k \right) \|x\|_2^2 \leq \|Ax\|_2^2 \leq \left( 1 + \delta_k \right) \|x\|_2^2 \]

Nonuniform recovery guarantees the recovery of a particular \( \text{(Fu04)} \quad \|A I \|_2 \leq \|A\|_2 \quad \|A \|_1 \quad \|x\|_1 \quad \|x\|_0 \quad \text{for all } x \in \mathbb{R}^n \]

while Fuchs, 2004 introduced the stronger conditions

\[ (\text{FU04a}) \quad \|A I \|_2 \leq \|A\|_2 \quad \|A\|_1 \quad \|x\|_1 \quad \|x\|_0 \quad \text{for all } x \in \mathbb{R}^n \]

(\text{FU04b}) is also known as a dual certificate or strong source condition (Möller, 2012):

\[ (\text{SSC}) \quad \exists p \in \mathbb{R}^n \quad \text{s.t.} \quad p \in \text{range}(A^T) \quad \|p\|_0 \leq k \]

Apart from its exact recovery guarantee, it also yields convergence rates and error estimates (e.g., Benning, 2011). The order between the conditions is given as

\[ \text{(Cho)} \Rightarrow \text{(Tr04)} \Rightarrow \text{(Fu04a)} \Rightarrow \text{(Fu04b)} \Rightarrow \text{(SSC)} \]

Conclusions

- For system matrices \( A \) from severely ill-posed inverse problems like EEG/MEG, conditions (Cho), (RIP), (Fu04a) might be too strong, even for a dense discretization.
- (Fu04b)/(SSC) are more difficult to compute, but may provide promising tools to analyze sparse recovery properties. In addition, they provide convergence rates and error estimates and extend to more general regularization (generalized) total variation (Benning, 2011, Möller, 2012) which have also been considered for EEG/MEG (Haufe et al., 2008, Gramfort et al., 2013).
- Note that we addressed spatial inversion only. Our results do not extend to temporal decoding in EEG/MEG!

Extensions and Outlook

- Going from normal constraint to vector reconstruction leads to block sparsity (see Haufe et al., 2008, Tellen, 2013).
- Neurophysiologically plausible source orientation constraints.
- Column-normalization is ambivalent in \( \ell_2 \)-norm approaches, situation for large \( n \) is unclear.
- The computation of (Fu04b)/(SSC) needs to be improved.
- Methodology needs to target clinically relevant questions to be meaningful in practice.
- Practical definition of the spatial resolution of sparse EEG/MEG.
- Examine EEG-MEG combination from a sparse inversion perspective.
- Incorporate noise, artifacts and non-sparse background activity

Table 1: Coherence of the EEG system matrix \( A \)

Table 2: Lower bound to \( \delta_k \) for Monte Carlo simulations

Table 3: Empirical probability of (Tr04) (left), (Fu04a)/(middle) and (Fu04b)/(SSC) (right) being true for \( k = 2 \) and \( N = 20000 \) samples.

Preliminary Results

Restriction to 3 head models, max \( n = 8000 \), EEG, scalar reconstruction.

- Table 1 shows the coherence of \( A \) which also upper-bounds \( \delta_k \), Table 2 shows lower bounds for \( \delta_k \) from extensive Monte Carlo simulations. For practice (\( n \geq 1000 \)) both extremely close to 1 and, thus, don't provide any recovery guarantees.

- Table 3 shows the empirical probabilities of the nonuniform conditions being true for \( k = 2 \) and \( N = 20000 \) samples. As expected, the likelihood rises from (Tr04) to (Fu04b)/(SSC).

- The gap between (Fu04a) and (Fu04b)/(SSC) is dramatic. Trends in (Fu04a) are not predictive of trends in (Fu04b)/(SSC).

- Results for MEG, other head models and/or for \( k > 3 \) confirm the results presented here. As expected, all guarantees degrade as the source density increases.

References


Figure 5: Procedure to build an individual, realistic, anisotropic finite element (FE) head model. Compartments: Skin, eyes, skull compacta, skull spongiosa, cal, grey and white matter of both cerebral and cerebellum and brain stem. For grey and white matter, anisotropic conductivity is used, which have been computed from diffusion weighted MRI (DMR) analysis. A detailed description is given in Tellen, 2013 and the references therein.

Figure 6: Preliminary to build an individual, realistic, anisotropic finite element (FE) head model. Compartments: Skin, eyes, skull compacta, skull spongiosa, cal, grey and white matter of both cerebral and cerebellum and brain stem. For grey and white matter, anisotropic conductivity is used, which have been computed from diffusion weighted MRI (DMR) analysis. A detailed description is given in Tellen, 2013 and the references therein.