When constructing 4x4 transformation matrices describing the relationship between local unprimed and primed coordinate systems, the translation vectors should be defined in the primed coordinate system (not the global or unprimed system). A 2D example is presented below.

The origin of the global system is \([0,0,0]\) and it has base vectors \([1,0,0]\), \([0,1,0]\) and \([0,0,1]\). The “fe” coordinate system has its origin (in the GLOBAL system) defined by the vector \(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\), and the “fm” coordinate system has its origin (in the GLOBAL system) defined by the vector \(\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}\). We wish to compute the 4x4 transformation matrix between the “fe” and “fm” coordinate systems, \(T_{fe \rightarrow fm}\).

For illustration purposes, we can write this transformation as:

\[
T_{fe \rightarrow fm} = T_{G \rightarrow fm} T_{fe \rightarrow G}
\]

The transformation \(T_{fe \rightarrow G}\) can be readily calculated, since the axes are aligned and there is only the translation to be defined:
Note that the translation vector between the \(fe\) and G coordinate systems, defined in the G coordinate system, is \(t_G^{fe} = [-1, 0, 0]\), the opposite of \(t_{fe}^G\). In this case, because the \(fe\) and G axes are aligned, there is no difference as to whether this vector is defined in the \(fe\) or G coordinate systems. Also note that the rotation matrix (the upper 3x3 portion of \(T_{fe \rightarrow G}\)) is simply the identity matrix because the axes of the two systems are aligned.

The transformation \(T_{G \rightarrow fm}\) must take into account that 1) there is a 90 degree counterclockwise rotation of the x-y axes between the G and fm systems, and 2) the translation vector must be defined in the fm coordinate system, which is rotated with respect to the G coordinate system:

\[
T_{G \rightarrow fm} = \begin{bmatrix}
0 & 1 & 0 & -2 \\
-1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Note that the translation vector between the G and fm coordinate systems, defined in the fm coordinate system, is \(t_{fm}^{G} = [-2, 0, 0]\). Think of it as a rotated version of the vector \(t_{G \rightarrow fm}^G = [1, -2, 0]\):

\[
\begin{align*}
t_{G \rightarrow fm}^G &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\
R_{G \rightarrow fm} \cdot t_{G \rightarrow fm}^G &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}
\end{align*}
\]

Now, the overall transformation matrix can be computed:

\[
T_{fe \rightarrow fm} = T_{G \rightarrow fm} T_{fe \rightarrow G} = \begin{bmatrix}
0 & 1 & 0 & -2 \\
-1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Examining the resulting translation vector, \(t_{fm}^{fe} = [-1, 0, 0]\). If we examine the figure above, we see that this makes sense as the \(fe\) and fm coordinate systems are separated by a translation of \(-1\) units along the x-axis of the fm system.

When constructing 4x4 transformation matrices between two local systems, you do not have to go to the trouble of using the intermediate global configuration as illustrated above, but you do need to make sure that your translation vectors are defined in the primed local system (fm in the case above).

For background, see [http://kwon3d.com/theory/transform/transform.html](http://kwon3d.com/theory/transform/transform.html), in particular equations (11) and (12).