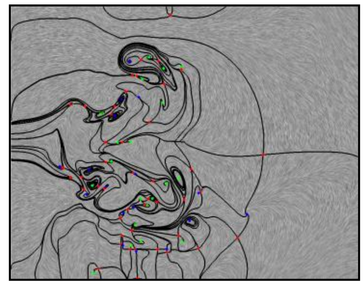
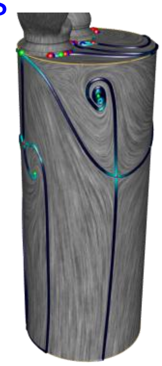
Vector Field Topology Continued

Examples





Motivation

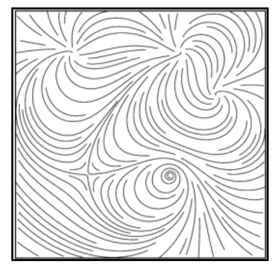
- Abstract representation of flow field
- Characterization of global flow structures
- Basic idea (steady case):
  - *Interpret flow in terms of streamlines*
  - *Classify them w.r.t. their limit sets*
  - *Determine regions of homogenous behavior*
- Graph depiction
- Fast computation

Limit Sets and Basins

- Limit sets of a point $\mathbf{x} \in \mathbb{R}^n$
  - $\omega(\mathbf{x})$: ***omega limit set of x*** = point (or curve) reached after ***forward*** integration by streamline seeded at x
  - $\alpha(\mathbf{x})$: ***alpha limit set of x*** = point (or curve) reached after ***backward*** integration by streamline seeded at x
- Sources (⟨) and sinks (⌉) of the flow
- Basin: region of influence of a limit set

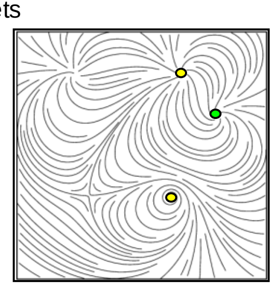
Limit Sets and Basins

- Phase portrait



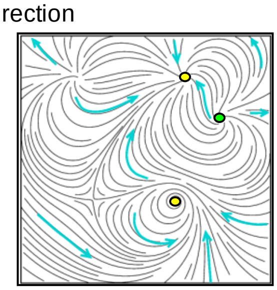
Limit Sets and Basins

- Limit sets



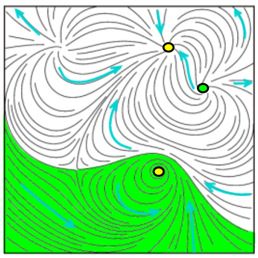
Limit Sets and Basins

- Flow direction



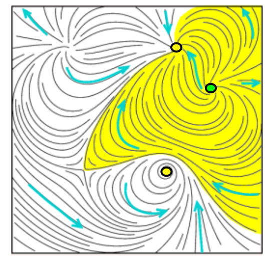
Limit Sets and Basins

- ⌉-basin of sink



Limit Sets and Basins

- 〈-basin of source

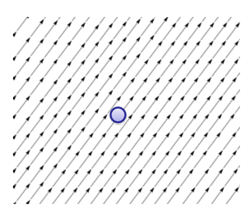


Critical Points

- Equilibrium
  - $\vec{v}(\mathbf{x}_0) = \vec{0}$
  - Streamline reduced to a single point
- Remarks
  - Asymptotic flow convergence / divergence
  - Streamlines “intersect” at critical points
- Type of critical point determines local flow pattern around it

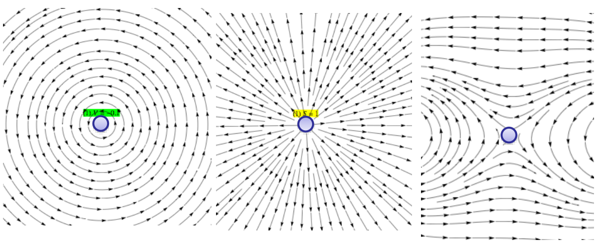
Intuition: Smooth Field

- In the ε–neighborhood of a regular point the direction of the vector field does not change significantly



Intuition: Smooth Field

- In the ε–neighborhood of a critical point the direction of the vector field can change arbitrarily



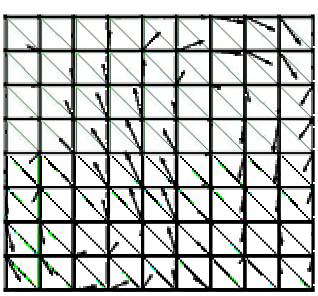
Critical Points are Key to Understand the Structure of the Field

- Computation as intersection of level sets:

$$\begin{cases} v_x(x,y) = 0 \\ v_y(x,y) = 0 \end{cases}$$

Triangular Mesh

- Walk through the mesh triangle by triangle



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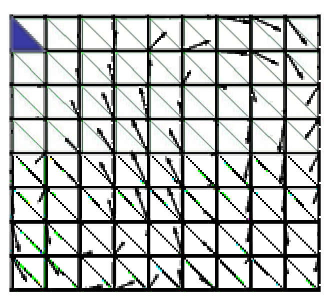

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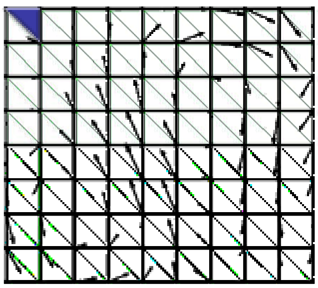

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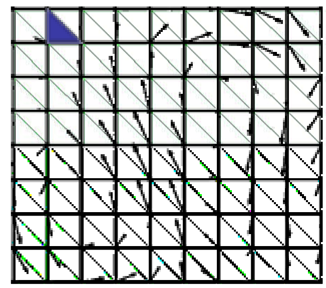

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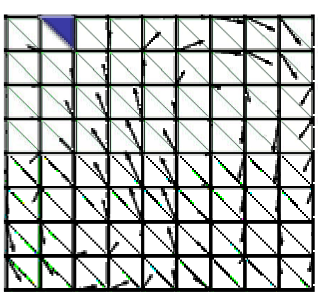
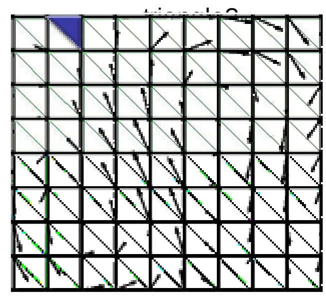


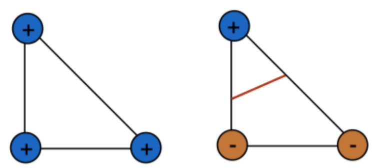
Triangular Mesh

- How many critical points can you have in a triangle?



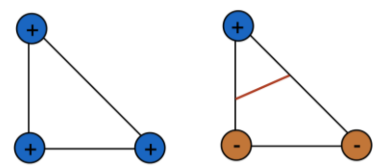
Remember Isocontours

$$v_x(x,y)=0$$



Remember Isocontours

$$v_y(x,y)=0$$



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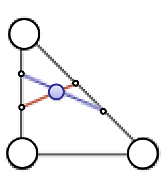

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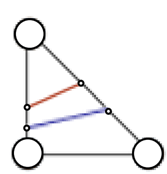
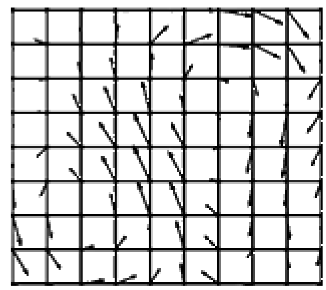


Remember Isocontours

- Walk around the boundary
- Alternating intersections with the two level sets:
  - One critical point
- Otherwise:
  - No critical point

Quad Mesh

- Walk through the mesh triangle by triangle



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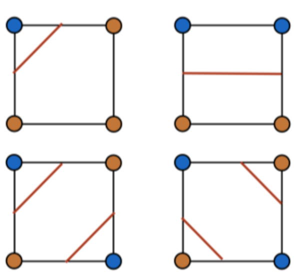

Remember Isocontours

- How many critical points per cell can you have?

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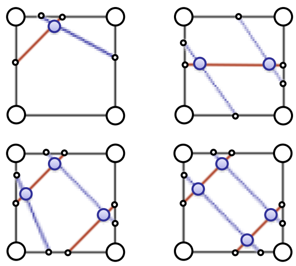

Remember Isocontours

- For each pair of connected components of Vx and Vy
  - Walk around the boundary
  - Alternating intersections with the two level sets:
    - One critical point
  - Otherwise:
    - No critical point

Critical Points

- Jacobian has **full rank** $\mathbf{J} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{pmatrix}$
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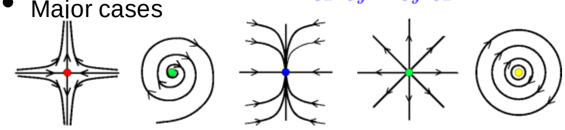
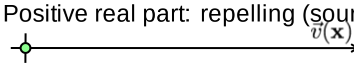


Saddle **Spiral** **Node** **Focus** **Center**

- Hyperbolic (stable) / non-hyperbolic (unstable) $\mathrm{Re}(\lambda_{1,2}) \neq 0$

Critical Points

- Type determined by Jacobian's eigenvalues:
  - Positive real part: repelling (source) $\vec{v}(\mathbf{x}) = k\mathbf{x},\ k > 0$



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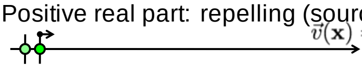

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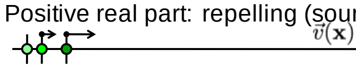

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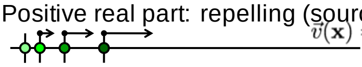

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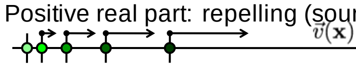

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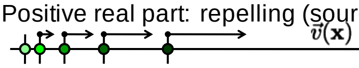

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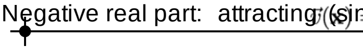

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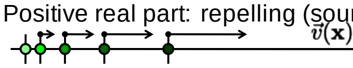

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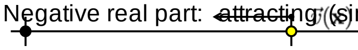

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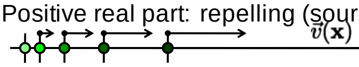

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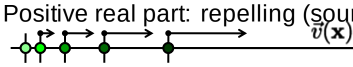

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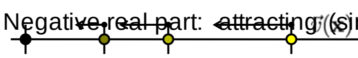

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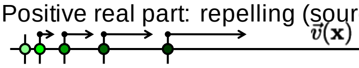

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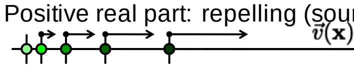

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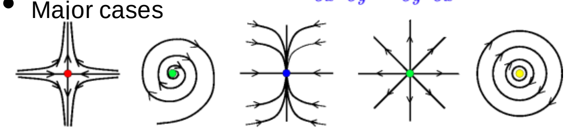
- Major cases



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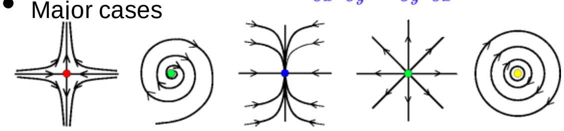

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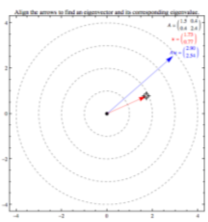
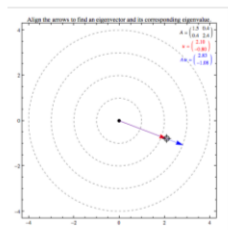
Eigenvalues and Eigenvectors

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad Ae = \lambda e$$





Eigenvalues and Eigenvectors

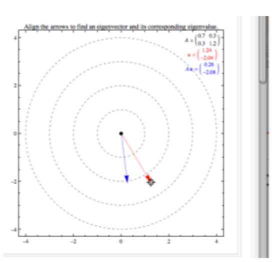
- Eigenvalues can be computed as the zeroes of the characteristic polynomial

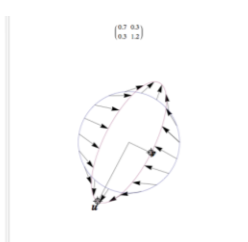
$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad - bc)$$

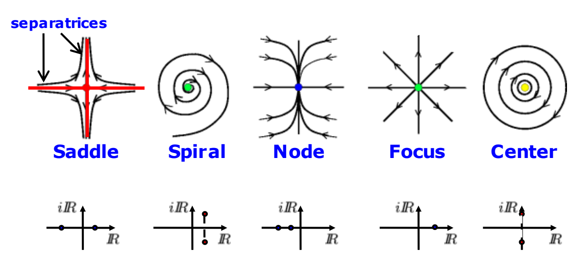
$$\lambda = \frac{a+d}{2} \pm \sqrt{\frac{(a+d)^2}{4} + bc - ad} = \frac{a+d}{2} \pm \frac{\sqrt{4bc + (a-d)^2}}{2}$$

Eigenvalues and Eigenvectors



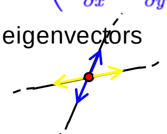


Critical Points



Critical Point Extraction

- Cell-wise analysis
  - Solve linear / quadratic equation to determine position of critical point in cell $\vec{v}(x,y)=\vec{0}$
  - Compute Jacobian at that position $\mathbf{J}=\begin{pmatrix}\frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y}\\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y}\end{pmatrix}$
  - Compute eigenvalues
  - If type is saddle, compute eigenvectors

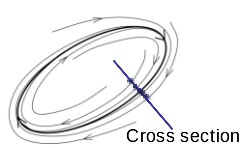


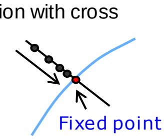
Closed Orbits

- Curve-type limit set
- Sink / source behavior

Closed Orbits

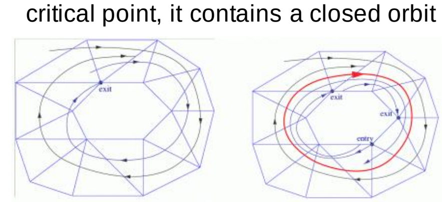
- Curve-type limit set
- Sink / source behavior
- Poincaré map:
  - Defined over cross section
  - Map each position to next intersection with cross section along flow
  - Discrete map
  - Cycle intersects at fixed point
  - Hyperbolic / non-hyperbolic





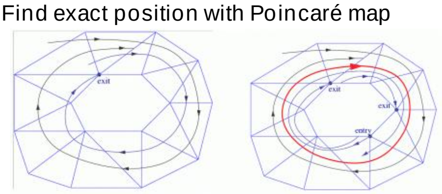
Closed Orbit Extraction

- Poincaré-Bendixson theorem:
  - If a region contains a limit set and no critical point, it contains a closed orbit



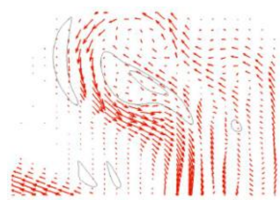
Closed Orbit Extraction

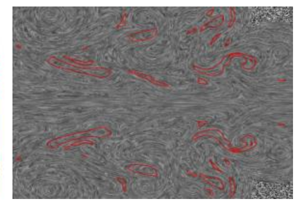
- Detect closed cell cycle
- Check for flow exit along boundary
- Find exact position with Poincaré map



Closed Orbit Extraction

- Results

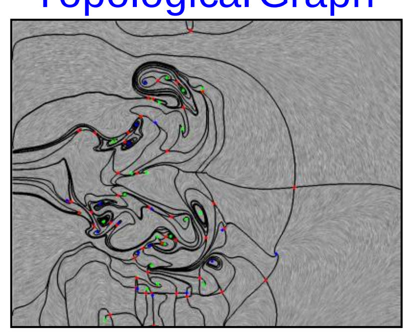




Topological Graph

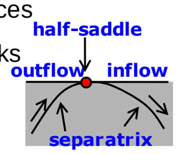
- Graph
  - Nodes: critical points
  - Edges: separatrices and closed orbits
- Remark
  - All streamlines in a given region have same ⟨- and ⌉-limit set
- Problem
  - Definition does not account for bounded domain

Topological Graph

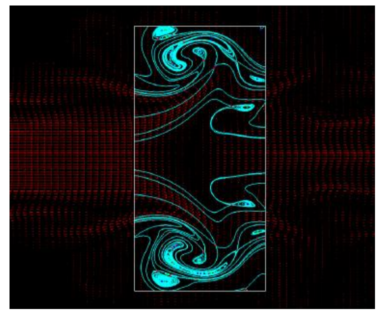


Local Topology

- Classification w.r.t. asymptotic convergence
- On bounded domain: streamlines leave domain in finite time
- Extend definition of topology
  - Inflow boundaries α sources
  - Outflow boundaries α sinks
  - Bounded by half-saddles
  - Additional separatrices



Local Topology



Application to Surfaces in 3D

http://people.nas.nasa.gov/~globus/topology/Pictures/pictures.html

- Critical point analysis + integration of separatrices applied to projection of vector field onto polygonal surface

What about transient flows?

- Parameter dependent topology:
- Critical points move, appear, vanish, transform
- Graph connectivity changes
- Structural stability (Peixoto): topology is stable w.r.t. small but arbitrary changes of parameter(s) if and only if
- 1) Number of critical points and closed orbits is finite and all are hyperbolic
- 2) No saddle-saddle connection