Vector Field Topology Continued
Examples
Motivation

- Abstract representation of flow field
- Characterization of global flow structures
- Basic idea (steady case):
  - Interpret flow in terms of streamlines
  - Classify them w.r.t. their limit sets
  - Determine regions of homogenous behavior
- Graph depiction
- Fast computation
Limit Sets and Basins

- Limit sets of a point $x \in \mathbb{R}^n$

- $\omega(x)$: **omega limit set of** $x$ = point (or curve) reached after *forward* integration by streamline seeded at $x$

- $\alpha(x)$: **alpha limit set of** $x$ = point (or curve) reached after *backward* integration by streamline seeded at $x$

- Sources (⟨) and sinks (⟩) of the flow

- Basin: region of influence of a limit set
Limit Sets and Basins

- Phase portrait
Limit Sets and Basins

- Limit sets
Limit Sets and Basins

- Flow direction
Limit Sets and Basins

- $\Lambda$-basin of sink
Limit Sets and Basins

- basin of source
Critical Points

• Equilibrium
  • \( \vec{v}(x_0) = \vec{0} \)
  • Streamline reduced to a single point

• Remarks
  • Asymptotic flow convergence / divergence
  • Streamlines “intersect” at critical points

• Type of critical point determines local flow pattern around it
Intuition: Smooth Field

- In the $\varepsilon$–neighborhood of a regular point the direction of the vector field does not change significantly
Intuition: Smooth Field

- In the $\varepsilon$–neighborhood of a critical point the direction of the vector field can change arbitrarily.
Critical Points are Key to Understand the Structure of the Field

- Computation as intersection of level sets:

\[
\begin{cases}
  v_x(x, y) = 0 \\
  v_y(x, y) = 0
\end{cases}
\]
Triangular Mesh

- Walk through the mesh triangle by triangle
Triangular Mesh

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- How many critical points can you have in a triangle?
Remember Isocontours

\[ v_x(x, y) = 0 \]
Remember Isocontours

\[ v_y(x, y) = 0 \]
Remember Isocontours

\[
\begin{aligned}
\nu_x(x, y) &= 0 \\
\nu_y(x, y) &= 0
\end{aligned}
\]
Remember Isocontours

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  v_x(x, y) = 0 \\
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\end{cases}
\]
Remember Isocontours

• Walk around the boundary
• Alternating intersections with the two level sets:
  • One critical point
• Otherwise:
  • No critical point
Quad Mesh

- Walk through the mesh triangle by triangle
Remember Isocontours
Remember Isocontours

- How many critical points per cell can you have?
Remember Isocontours

- How many critical points per cell can you have?
Remember Isocontours

- For each pair of connected components of $V_x$ and $V_y$
- Walk around the boundary
- Alternating intersections with the two level sets:
  - One critical point
- Otherwise:
  - No critical point
Critical Points

- Jacobian has **full rank**
- No zero eigenvalue
- Major cases

\[ J = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{pmatrix} \]

\[ \det J = \frac{\partial v_x}{\partial x} \frac{\partial v_y}{\partial y} - \frac{\partial v_x}{\partial y} \frac{\partial v_y}{\partial x} \equiv \lambda_0 \lambda_1 \neq 0 \]

- **Saddle**
- **Spiral**
- **Node**
- **Focus**
- **Center**

- Hyperbolic (stable) / non-hyperbolic (unstable)

\[ \text{Re}(\lambda_{1,2}) \neq 0 \]
Critical Points

• Type determined by Jacobian’s eigenvalues:

• Positive real part: repelling (source)

\[ \vec{v}(x) = kx, \ k > 0 \]
Critical Points

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  • Negative real part: attracting \( \vec{v}(x) = kx, \ k < 0 \)
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Critical Points

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  • Positive real part: repelling (source) \( \vec{v}(x) = kx, \ k > 0 \)

  ![Diagram of positive real part]

  • Negative real part: attracting \( \vec{v}(x) = kx, \ k < 0 \)

  ![Diagram of negative real part]
Critical Points

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  - Positive real part: repelling (source) \( \ddot{v}(x) = kx, \ k > 0 \)
  - Negative real part: attracting \( \ddot{v}(x) = kx, \ k < 0 \)
Critical Points

- Type determined by Jacobian's eigenvalues:
  - Positive real part: repelling (source) $\vec{v}(x) = kx, \ k > 0$
  - Negative real part: attracting (sink) $\vec{v}(x) = kx, \ k < 0$
Critical Points

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- Major cases

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Critical Points

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• Major cases

  Saddle  Spiral  Node  Focus  Center

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Eigenvalues and Eigenvectors

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad A e = \lambda e \]
Eigenvalues and Eigenvectors

- Eigenvalues can be computed as the zeroes of the characteristic polynomial

\[ \det(A - \lambda I) = 0 \]

\[
\begin{vmatrix}
  a - \lambda & b \\
  c & d - \lambda 
\end{vmatrix}
= (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc)
\]

\[
\lambda = \frac{a + d}{2} \pm \sqrt{\frac{(a + d)^2}{4} + bc - ad} = \frac{a + d}{2} \pm \sqrt{\frac{4bc + (a - d)^2}{2}}
\]
Eigenvalues and Eigenvectors

Align the arrows to find an eigenvector and its corresponding eigenvalue.

\[
A = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 1.2 \end{pmatrix}, \quad u = \begin{pmatrix} 1.26 \\ -2.08 \end{pmatrix}, \quad Au = \begin{pmatrix} 0.26 \\ -2.08 \end{pmatrix}
\]
Critical Points

Saddle

Spiral

Node

Focus

Center

separatrices
Critical Point Extraction

- Cell-wise analysis
- Solve linear / quadratic equation to determine position of critical point in cell \( \vec{v}(x_0, y_0) = \vec{0} \)
- Compute Jacobian at that position: 
  \[ J = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{pmatrix} \]
- Compute eigenvalues
- If type is saddle, compute eigenvectors
Closed Orbits

- Curve-type limit set
- Sink / source behavior
Closed Orbits

- Curve-type limit set
- Sink / source behavior
- Poincaré map:
  - Defined over cross section
  - Map each position to next intersection with cross section along flow
  - Discrete map
  - Cycle intersects at fixed point
  - Hyperbolic / non-hyperbolic
Closed Orbit Extraction

- Poincaré-Bendixson theorem:
  - If a region contains a limit set and no critical point, it contains a closed orbit
Closed Orbit Extraction

- Detect closed cell cycle
- Check for flow exit along boundary
- Find exact position with Poincaré map
Closed Orbit Extraction

• Results
Topological Graph

- **Graph**
  - Nodes: critical points
  - Edges: separatrices and closed orbits

- **Remark**
  - All streamlines in a given region have same \(-\) and \(\bar{\lim}\)-limit set

- **Problem**
  - Definition does not account for bounded domain
Topological Graph
Local Topology

• Classification w.r.t. asymptotic convergence
• On bounded domain: streamlines leave domain in finite time
• Extend definition of topology
  • Inflow boundaries $\alpha$ sources
  • Outflow boundaries $\alpha$ sinks
  • Bounded by half-saddles
  • Additional separatrices
Local Topology
Critical point analysis + integration of separatrices applied to projection of vector field onto polygonal surface

http://people.nas.nasa.gov/~globus/topology/Pictures/pictures.html