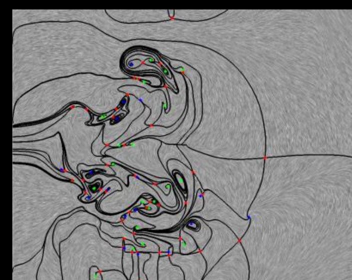
Vector Field Topology

Thanks to Xavier Tricoche

Examples




Motivation

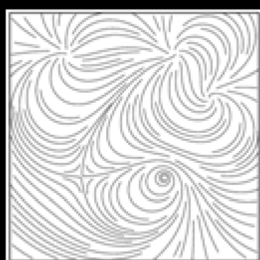
- Abstract representation of flow field
- Characterization of global flow structures
- Basic idea (steady case):
  - *Interpret flow in terms of streamlines*
  - *Classify them w.r.t. their limit sets*
  - *Determine regions of homogenous behavior*
- Graph depiction
- Fast computation

Limit Sets and Basins

- Limit sets of a point $\mathbf{x} \in \mathbb{R}^n$
  - $\omega(\mathbf{x})$ *: **omega limit set of x** =* *point (or curve) reached after **forward** integration by streamline seeded at x*
  - $\alpha(\mathbf{x})$ *: **alpha limit set of x** =* *point (or curve) reached after **backward** integration by streamline seeded at x*
- Sources (α) and sinks (ω) of the flow
- Basin: region of influence of a limit set

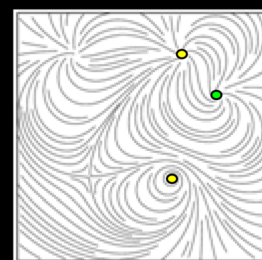
Limit Sets and Basins

- Phase portrait



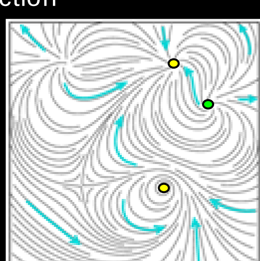
Limit Sets and Basins

- Limit sets



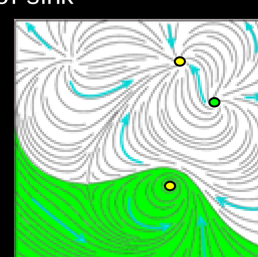
Limit Sets and Basins

- Flow direction



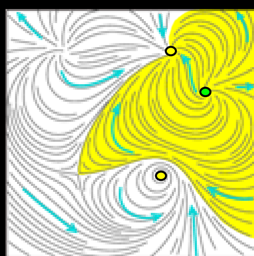
Limit Sets and Basins

- ω-basin of sink



Limit Sets and Basins

- α-basin of source



Critical Points

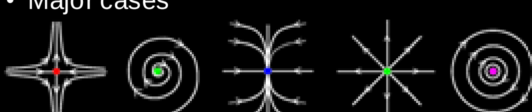
- Equilibrium
  - $\vec{v}(\mathbf{x}_0) = \vec{0}$
  - *Streamline reduced to a single point*
- Remarks
  - *Asymptotic flow convergence / divergence*
  - *Streamlines "intersect" at critical points*
- Type of critical point determines local flow pattern around it

Critical Points

- Jacobian has full rank
  - *No zero eigenvalue*

$$Df(x_0) = \begin{pmatrix} \frac{df_1}{dx_1} & \cdots & \frac{df_1}{dx_n} \\ \vdots & & \vdots \\ \frac{df_n}{dx_1} & \cdots & \frac{df_n}{dx_n} \end{pmatrix}$$

- Major cases



- Hyperbolic / non-hyperbolic

Critical Points

- Type determined by Jacobian's eigenvalues:
  - *Positive real part: repelling (source)*

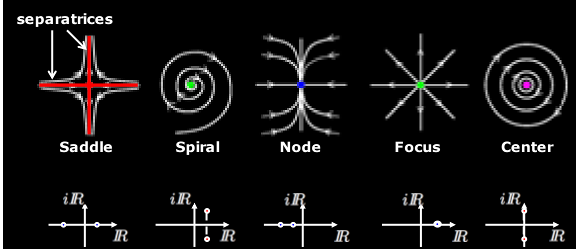
$\vec{v}(\mathbf{x}) = k\mathbf{x},\ k > 0$

  - *Negative real part: attracting (sink)*

$\vec{v}(\mathbf{x}) = k\mathbf{x},\ k < 0$

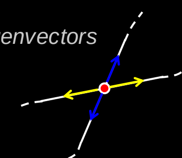
  - *Complex: rotation*

Critical Points



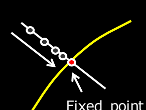
Critical Point Extraction

- Cell-wise analysis
  - *Solve linear / quadratic equation to determine position of critical point in cell*
  - *Compute Jacobian at that position*
  - *Compute eigenvalues*
  - *If type is saddle, compute eigenvectors*



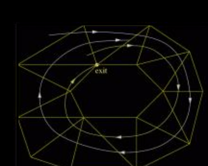
Closed Orbits

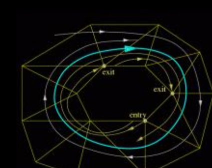
- Curve-type limit set
- Sink / source behavior
- Poincaré map:
  - *Defined over cross section*
  - *Map each position to next intersection with cross section along flow*
  - *Discrete map*
  - *Cycle intersects at fixed point*
  - *Hyperbolic / non-hyperbolic*



Closed Orbit Extraction

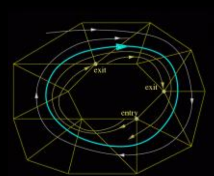
- Poincaré-Bendixson theorem:
  - *If a region contains a limit set and no critical point, it contains a closed orbit*





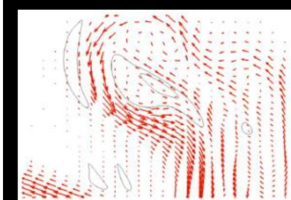
Closed Orbit Extraction

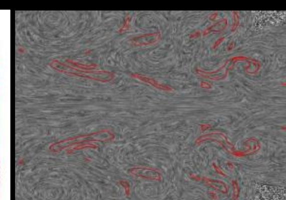
- Detect closed cell cycle
- Check for flow exit along boundary
- Find exact position with Poincaré map



Closed Orbit Extraction

- Results





Topological Graph

- Graph
  - *Nodes: critical points*
  - *Edges: separatrices and closed orbits*
- Remark
  - *All streamlines in a given region have same α- and ω-limit set*
- Problem
  - *Definition does not account for bounded domain*

Topological Graph