Quick Review of Eigenvalues and Eigenvectors

Consider an $n \times n$ symmetric real matrix M. The vector $\mathbf{e}$ is an eigenvector with eigenvalue λ if

$$M\mathbf{e} = \lambda\mathbf{e}$$

Note that if $\mathbf{e}$ is an eigenvector then so is $k\mathbf{e}$ for any non-zero k. Hence the magnitude of these vectors is of no significance, and we may as well use unit vectors when it is useful to do so.

To solve the above equation for the eigenvectors and eigenvalues of the matrix M, we can rewrite it in the form

$$(M - \lambda I)\mathbf{e} = \mathbf{0}$$

where I is the $n \times n$ identity matrix and $\mathbf{0}$ is a vector of zeroes.

This set of linear equations is homogeneous since the constant term in each equation is zero. If we can invert $(M - \lambda I)$, then we can multiply this inverse by $\mathbf{0}$ to obtain the "trivial" solution $\mathbf{0}$ for $\mathbf{e}$. The only way non-zero solutions are possible is if the matrix is singular, that is, when

$$\det(M - \lambda I) = 0$$

The determinant can be written as a sum of terms, each of which is a product of matrix elements, one from each row. Since λ occurs in the elements on the diagonal, terms of up to order n in λ may occur in these products. So the determinant is a polynomial of order n in λ. This polynomial will have n roots. We will assume that these roots are distinct.

As a simple illustrations, consider the 2×2 real symmetric matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Its eigenvalues can be found by finding the roots of

$$\det\begin{pmatrix} a-\lambda & b \\ b & c-\lambda \end{pmatrix} = 0$$

that is,

$$\lambda^2 - (a+c)\lambda + (ac - b^2) = 0$$

The two roots are,

$$\lambda_{+,-} = \frac{(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}{2}$$

or, if we let $d = \sqrt{(a-c)^2 + 4b^2}$, then

$$\lambda_{+,-} = \frac{(a+c) \pm d}{2}$$

If λ_+ and λ_- are the values with the $+$ and $-$ sign respectively in front of the square-root, then $\lambda_+ > \lambda_-$.

Corresponding to each eigenvalue λ_i will be an eigenvector $\mathbf{e}_i$ obtained by solving the set of n homogeneous linear equations

$$(M - \lambda_i I)\mathbf{e}_i = 0$$

One way to find an eigenvector is to note that it must be orthogonal to each of the rows of the matrix $(M - \lambda_i I)$.

Again, as a simple illustration, consider the 2×2 real symmetric matrix above. We have to solve

$$\begin{pmatrix} a-\lambda & b \\ b & c-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

A vector orthogonal to the first row is given by

$$\begin{pmatrix} -b \\ a-\lambda \end{pmatrix}$$

The two eigenvectors can now be determined by substituting the values λ_+ and λ_- determined above. Similarly, a vector orthogonal to the second row is

$$\begin{pmatrix} c-\lambda \\ -b \end{pmatrix}$$

The two apparently different directions for the eigenvector are actually the same if λ is an eigenvalue, since then $b/(a-\lambda) = (c-\lambda)/b$.

The eigenvectors can thus be written in either of the forms

$$\begin{pmatrix} 2b \\ (c-a) \pm d \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} (a-c) \pm d \\ 2b \end{pmatrix}$$

The magnitude squared of the first form is $2d(d \pm (c-a))$, while the magnitude squared of the second is $2d(d \pm (a-c))$. The unit eigenvectors can thus be written in either of the forms:

$$\frac{1}{\sqrt{2d}\sqrt{d \pm (c-a)}} \begin{pmatrix} \pm 2b \\ d \pm (c-a) \end{pmatrix}$$

or

$$\frac{1}{\sqrt{2d}\sqrt{d \pm (a-c)}} \begin{pmatrix} d \pm (a-c) \\ \pm 2b \end{pmatrix}$$

In each occurence of $\pm$ the plus sign is chosen when the plus sign is chosen in the expression for the eigenvalue, while the minus sign is chosen when the minus sign is chosen in the expression for the eigenvalue. Any non-zero multiple of these vectors is, of course, also an eigenvector (Note that the two forms happen to give opposite directions when $b < 0$).