Applying Constraints to the Electrocardiographic Inverse Problem

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Electrocardiography
**Electrocardiographic Mapping**

- Bioelectric Potentials
- Goals
  - Higher spatial density
  - Imaging modality
- Measurements
  - Body surface
  - Heart surfaces
  - Heart volume

**Body Surface Potential Mapping**

Taccardi et al, Circ., 1963
Cardiac Mapping

- Coverage
- Sampling Density
- Surface or volume

Inverse Problems in Electrocardiography

Forward

Inverse
Epicardial Inverse Problem

- Definition
  - Estimate sources from remote measurements

- Motivation
  - Noninvasive detection of abnormalities
  - Spatial smoothing and attenuation

Forward/Inverse Problem

Forward problem

- Epicardial/Endocardial Activation Time
- Geometric Model
- Body Surface Potentials

Inverse problem

Thom Oostendorp, Univ. of Nijmegen
Sample Problem: PTCA
Elements of the Inverse Problem

- Components
  - Source description
  - Geometry/conductivity
  - Forward solution
  - “Inversion” method (regularization)
- Challenges
  - Inverse is ill-posed
  - Solution ill-conditioned

Inverse Problem Research

- Role of geometry/conductivity
- Numerical methods
- Improving accuracy to clinical levels
- Regularization
  - A priori constraints versus fidelity to measurements
Regularization

- Current questions
  - Choice of constraints/weights
  - Effects of errors
  - Reliability
- Contemporary approaches
  - Multiple Constraints
  - Time Varying Constraints
  - Tuned constraints
  - Multisource constraints

Tikhonov Approach

Problem formulation

\[ y(k) = A \cdot h(k) + e(k) \quad k = 1, 2, \ldots, L \]

Constraint

\[ \hat{h}_\lambda = \arg \min_x \left( \|y - Ax\|^2 + \lambda^2 \|Rx\|^2 \right) , \]

Solution

\[ \hat{h}_\lambda = (A^T A + \lambda^2 R^T R)^{-1} A^T y \]
Multiple Constraints

For $k$ constraints

$$\hat{h}_\lambda = \arg\min_x \left( \|y - Ax\|^2 + \sum_{i=1}^k \lambda_i^2 \|R_i x\|^2 \right)$$

with solution

$$\hat{h}_\lambda = \left( A^T A + \sum_{i=1}^k \lambda_i^2 R_i^T R_i \right)^{-1} A^T y.$$  

Note: two regularization factors required

Dual Spatial Constraints

For two spatial constraints:

$$\hat{h}_\lambda = \left( A^T A + \lambda_1^2 R_1^T R_1 + \lambda_2^2 R_2^T R_2 \right)^{-1} A^T y.$$  

Note: two regularization factors required
Joint Time-Space Constraints

Redefine \( y, h, A \):

\[
\bar{y} = \bar{A} h + \bar{e}
\]

\[
\bar{A} = \begin{pmatrix}
A & 0 & 0 & \cdots & 0 \\
0 & A & 0 & \cdots & 0 \\
0 & 0 & A & \cdots & 0 \\
\vdots & & & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A
\end{pmatrix}
\]

And write a new minimization equation:

\[
\hat{h} = \arg \min_{\bar{x}} \left( \| \bar{A} \bar{x} - \bar{y} \|^2 + \sum_{i=1}^{k_x} \lambda_i^2 \| \bar{R}_i \bar{x} \|^2 + \sum_{i=1}^{k_t} \eta_i^2 \| \bar{T}_i \bar{x} \|^2 \right).
\]

Joint Time-Space Constraints

General solution:

\[
\hat{h} = \left( \bar{A}^T \bar{A} + \sum_{i=1}^{k_x} \lambda_i^2 \bar{R}_i^T \bar{R}_i + \sum_{i=1}^{k_t} \eta_i^2 \bar{T}_i^T \bar{T}_i \right)^{-1} \bar{A}^T \bar{y}
\]

For a single space and time constraint:

\[
\hat{h} = \left( \bar{A}^T \bar{A} + \lambda^2 \bar{R}^T \bar{R} + \eta^2 \bar{T}^T \bar{T} \right)^{-1} \bar{A}^T \bar{y}
\]

\[
= \left[ \bar{I}_L \otimes (\bar{A}^T \bar{A}) + \lambda^2 \bar{I}_L \otimes \bar{R}^T \bar{R} + \eta^2 (\bar{T}^T \bar{T}) \otimes \bar{I}_N \right]^{-1} \cdot (\bar{I}_L \otimes \bar{A}^T) \bar{y}.
\]

Note: two regularization factors and implicit temporal factor
Determining Weights

- Based on *a posteriori* information
- *Ad hoc* schemes
  - CRESO: composite residual and smooth operator
  - BNC: bounded norm constraint
  - AIC: Akaike information criterion
  - L-curve: residual norm vs. solution seminorm

**L-Surface**

- Natural extension of single constraint approach
- “Knee” point becomes a region
Joint Regularization Results

Energy Regularization Parameter

Laplacian Regularization Parameter

RMSE

with Fixed Laplacian Parameter

with Fixed Energy Parameter

Admissible Solution Approach

Constraint 1 (non-differentiable but convex)
Constraint 2 (non-differentiable but convex)
Constraint 3 (differentiable)
Constraint 4 (differentiable)

Admissible Solution Region
**Single Constraint**

Define $\phi(x)$ s.t.

$$\phi(x) : \mathcal{R}^N \rightarrow \mathcal{R}$$

with the constraint such that

$$\phi(x) - \epsilon < 0.$$ 

that satisfies the convex condition

$$\phi(\alpha x + (1 - \alpha)y) \leq \alpha \phi(x) + (1 - \alpha)\phi(y) \quad \forall \alpha \in [0, 1].$$

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**Multiple Constraints**

Define multiple constraints $\phi_i(x)$

$$(\phi_i(x) - \epsilon_i) \in \mathcal{H}, \text{ for } i = 1, 2, \ldots, m.$$ 

so that the set of these

$$\{x : \phi(x) < 0\}$$

represents the intersection of all constraints. When they satisfy the joint condition

$$\phi(x) \leq 0$$

Then the resulting $x$ is the *admissible solution*
Examples of Constraints

- Residual constraint: \( \phi(x) = \|Ax - y\|_2^2 \)
- Regularization constraints: \( \phi(x) = \|Rx\|_2^2 \)
- Tikhonov constraints: \( \phi_\lambda(x) = \| (A \sqrt{\lambda R}) x - \begin{pmatrix} b \\ 0 \end{pmatrix} \|_2^2 \)
- Spatiotemporal constraints
- Weighted constraints
- Novel constraints

Admissible Solution Results

Original  Regularized  Admissible Solution
Combining Information Sources

Venous Catheter Based Mapping
Statistical Estimation

Training Data

Sparse Test Data

Covariance Matrix

Estimation Matrix

\[ E = C_{ku}^{T} C_{kk}^{-1} \]

\[ t_u = E t_k \]

Estimated Activation Maps

- Training set composition
- Lead selection
Augmented Inverse Problem

Torso geometry
+ Body-Surface Potentials
+ Sparse Epicardial Potentials
+ Inverse Solution

Epicardial Map

Subtraction Approach

Unknown

1) Subtract known epicardial potentials
2) Solve reduced inverse problem

Known

Inverse (Tikhonov)
Epicardial Estimation

Combined Estimation
Bayesian Approach

Hybrid Approach
Tank/Heart Geometry

Test Lead Sets

Anterior
- 42 leads
- 21 leads
- 10 leads

Posterior
Simulation Study

- 490 lead measured sock data
- Surrogate catheter potentials
  - 42 sites
  - + Gaussian noise
- Torso potentials
  - Calculated noise-free using forward model
  - + Gaussian noise

Leave-One-Experiment Out Protocol

<table>
<thead>
<tr>
<th>Training Data</th>
<th>Test Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>LV Paced Beats</td>
<td>Dec 18, 2000</td>
</tr>
<tr>
<td>Mixed Paced Beats</td>
<td>Dec 13, 2000</td>
</tr>
</tbody>
</table>
LV Pacing (LV-23 ms)

LV Pacing (LV-38 ms)
LV Pacing (LV-47 ms)

- Orig
- Subt
- MAP
- Epi Est
- Comb Est
- Hybrid MAP

31: LV-MEPiP-47 ms

LV Pacing (Mixed-23 ms)

- Orig
- Subt
- MAP
- Epi Est
- Comb Est
- Hybrid MAP

31: LV-RV-MEPiP-23 ms
**Estimation Findings**

- Estimation alone: noisy, unstable results
- Estimation + inverse: smoothing improves stability

**Inverse Solution Findings**

- All solutions better than simple Tikhonov
- MAP usually improved with addition of catheter measurements (Hybrid MAP)
Role of Statistics (Training)

• Generally helps
• But can add artifacts, e.g., spurious breakthroughs or wavefronts
• Torso potentials can reduce artifacts

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