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INSTITUT FÜR ANGEWANDTE MECHANIK

A Short Course on Boundary Element Methods

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1 Introduction

Engineers who are familiar with finite elements very often ask why it is necessary to develop yet another computational technique. The answer is that finite elements have been proved to be inadequate or inefficient in many engineering applications and what is perhaps more important is in many cases cumbersome to use and hence difficult to implement in Computer Aided Engineering systems. Finite Element (FE) analysis is still a comparatively slow process due to the need to define and redefine meshes in the piece or domain under study.

Boundary elements (BE) have emerged as a powerful alternative to finite elements particularly in cases where better accuracy is required due to problems such as stress concentration or where the domain extends to infinity. The most important feature of boundary elements, however, is that different to the finite domain methods as, e.g., the finite difference method or the finite element method, the methodology of formulating boundary value problems as boundary integral equations describes problems only by equations with known and unknown boundary states.Hence, it only requires discretization of the surface rather than the volume, i.e., the dimension of problems is reduced by one. Consequently, the necessary discretization effort is mostly much smaller and , moreover, meshes can easily be generated and design changes do not require a complete remeshing.

The BE method is especially advantageous in the case of problems with infinite or semi-infinite domains, e.g., so-called exterior domain problems: there, although only the finite surface of the infinite domain has to be discretized, the solution at any arbitrary point of the domain can be found after determining the unknown boundary data.

To be objective, the features of the BE method should be compared to its main rival, the FE method. Its advantages and disadvantages can be summarized as follows

1.1 Advantages of the Boundary Element Method

1. *Less data preparation time*: This is adirect result of the 'surface-only' modelling. Thus, the analyst's time required for data preparation and data checking for a given problem should be greatly reduced. Furthermore, subsequent changes in meshes are made easier.

2. *High resolution of stress*: Stresses are accurate because no further approximation is imposed on the solution at interior points, i.e., solution is exact and fully continuous inside the domain.

3. *Less computer time and storage*: For the same level of accuracy, the BE method uses a lesser number of nodes and elements (but a fully populated matrix), i.e., to achieve comparable accuracy in stress values, FE meshes would need more boundary divisions than the equivalent BE meshes.

4. *Less unwanted information*: In most engineering problems, the 'worst' situation (such as fracture, stress concentration, thermal shocks a.s.o.) usually occur on the surface.

Thus, modelling an entire three-dimensional body with finite elements and calculating stress (or other states) at every nodal point is very inefficient because only a few of these values will be incorporated in the design analysis. Therefore, using boundary elements is a very effective use of computing resources, and, furthermore, since internal points in BE solutions are optional, the user can focus on a particular interior region rather than the whole interior.

1.2 Disadvantages of the BE method

1. *Unfamiliar mathematics*: The mathematics used in BE formulations may seem unfamiliar to engineers (but not difficult to learn). However, many FE numerical procedures are directly applicable to BE solutions (such as numerical integration, surface approximation, treatment of boundary conditions).

2. *In non-linear problems, the interior must be modelled*: Interior modelling is unavoidable in non-linear material problems. However, in many non-linear cases (such as elastoplasticity) interior modelling can be restricted to selected areas such as the region around a crack tip.

3. *Fully populated and unsymmetric solution matrix*: The solution matrix resulting from the BE formulation is unsymmetric and fully populated with non-zero coefficients, whereas the FE solution matrices are usually much larger but sparsely populated. This means that the entire BE solution matrix must be saved in the computer core memory. However, this is not a serious disadvantage because to obtain the same level of accuracy as the FE solution, the BE method needs only a relatively modest number of nodes and elements.

4. *Poor for thin structures (shell) three-dimensional analyses*: This is because of the large surface/volume ratio and the close proximity of nodal points on either side of the structure thickness. This causes inaccuracies in the numerical integrations.

1.3 Choosing BE or FE?

To decide whether BE or FE solutions are more suitable for a particular problem, three factors must be taken into consideration:

1. The type of problem (linear, non-linear, shell-like analysis, etc.)
2. The degree of accuracy required
3. The amount of time to be spent in preparing and interpreting data.

Both techniques should be made available to engineers, because in certain types of applications one of them may display a distinct advantage over the other. Considering the advantages and disadvantages of the BE method listed above, the following points may help in deciding which technique to use:

a) The BE method is suitabable and more accurate for linear problems, particularly for three-dimensional problems with rapidly changing variables such as fracture or contact problems

b) Because of the much reduced time needed to model a particular problem, the BE method is suitable for preliminary design analyses where geometry and loads can be subsequently modified with minimal effort. This gives designers more freedom in experimenting with new shapes and geometries.

c) The FE method is more established and more commercially developed, particularly for complex non-linear problems where thorough tests to establish its reliability have been performed. The temptatipn for engineers is to use a well-established computer program rather than venture into new methods.

d) Mesh generators and plotting routines developed for FE applications are directly applicable to BE problems. It should not be a difficult task to write 'translator' programs to interface with commercial FE packages. Furthermore, many load incrementation and iterative routines developed for FE applications in non-linear problems are also directly applicable in BE algorithms.

2 Mathematical Preliminaries

For an easy understanding of the boundary integral equation derivation, some mathematical techniques are important. They will be used time and time again to transform the differential equations governing continuum mechanic problems into equivalent boundary integral equations. Moreover, some notations, definitions and useful formulas should be familiar to the reader in order to feel confident about their subsequent use. Proofs for these formulas and results can be found in textbooks on calculus and analysis.

2.1 Some notations and definitions

Here, the notations used in the following text are introduced and some definitions are given.

2.1.1 Indicial and symbolic notation

The components of a tensor of any order may be represented clearly by the use of the *indicial notation,* i.e., letter indices as subscripts are appended to the *generic* letter representing the tensor quantity of interest. Dependent on the number these indices, a tensor of *first order*, mostly called *vector,* bears one free index, a *second-order* tensor, sometimes called *dyadic*, has two free indices, a.s.o. Hence, a symbol such as λ which has no indices attached, represents a *scalar* or tensor of zero order.

When an index appears *twice* in a term, that index is understood to take on all the values of its range, and the resulting terms *summed.* In this so-called *Einstein summation convention*, repeated indices are often referred to as *dummy* indices, since their replacement by any other letter not appearing as a free index does not change the meaning of the term in which they occur. In ordinary physical space, the range of the indices is $1, 2, 3$.

The representation of a vector and a tensor in the *symbolic notation* is designated by bold-faced letters, e.g., $\mathbf{a}$ and $\mathbf{D}$, respectively, where unit vectors $\hat{\mathbf{e}}_i$ are further distinguished by a caret placed over the bold-faced letter. There, the summation convention is often also employed in connection with indexed base vectors $\hat{\mathbf{e}}_i$, i.e., for a vector

$$\mathbf{v} = v_i\hat{\mathbf{e}}_i = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3 \tag{2.1}$$

and similarly for an arbitrary dyadic

$$\mathbf{D} = D_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \tag{2.2}$$

A special operational vector is ∇, the so-called *Nabla vector,* containing differentiations with respect to all coordinate axis, e.g., for Cartesian coordinates x_1, x_2, x_3

$$\nabla = \frac{\partial}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{\partial}{\partial x_3}\hat{\mathbf{e}}_3 = \frac{\partial}{\partial x_i}\hat{\mathbf{e}}_i \triangleq \partial_i \tag{2.3}$$

2.1.1.1 Exercise 1: Nabla vector

Derive the Nabla vector for the Polar coordinates r and φ where the relations between the unit vectors $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\varphi$ of the Polar coordinate system and the unit vecors $\hat{\mathbf{e}}_1$and $\hat{\mathbf{e}}_2$ of theCartesian coordinate system are

$$\begin{aligned}\hat{\mathbf{e}}_1 &= \hat{\mathbf{e}}_r \cos\varphi - \hat{\mathbf{e}}_\varphi \sin\varphi \\ \hat{\mathbf{e}}_2 &= \hat{\mathbf{e}}_r \sin\varphi + \hat{\mathbf{e}}_\varphi \cos\varphi\end{aligned}$$

2.1.2 Contraction and different products of tensors

The *outer product* of two tensors is the tensor whose components are formed by multiplying each component of one of the tensors by every component of the other, i.e., a dyad is formed from two vectors by this very product

$$\begin{array}{ll}\text{Indicial Notation} & \text{Symbolic Notation} \\ a_i b_j = T_{ij} & \mathbf{ab} = \mathbf{T}\end{array} \tag{2.4}$$

where the symbols a_i and $b_j$can be in any order. Also, one obtains, e.g.,

$$\sigma_{ij} n_k = s_{ijk}, \quad \sigma_{ij}\varepsilon_{kl} = E_{ijkl}$$

Contraction of a tensor with respect to two free indices is the operation of assigning to both indices the same letter subscript, i.e., changing them to dummy indices, and, hence, performing the summation convention, e.g.,

$$T_{ii} = T_{11} + T_{22} + T_{33} = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i$$

$$\sigma_{ij} n_j = p_i$$

An *inner product* or *scalar product* of two tensors of arbitrary order is the result of a contraction, involving one index from each tensor, performed on the outer product of the two tensors, e.g.,

$$\begin{array}{ll}\text{Indicial Notation} & \text{Symbolic Notation} \\ a_i b_i = \lambda & \mathbf{a}\cdot\mathbf{b} = \lambda \\ D_{ij} n_j = p_i & \mathbf{D}\cdot\mathbf{n} = \mathbf{p} \\ D_{ij} n_i = f_j & \mathbf{n}\cdot\mathbf{D} = \mathbf{f}\end{array} \tag{2.5}$$

Example: The *directional derivative* of a scalar function $f(x_1, x_2, x_3)$ in the direction of a unit vector, e.g., the unit normal vector $\mathbf{n}$ is defined by the scalar product of this unit vector $\mathbf{n}$ with grad$f = \nabla f$:

$$\frac{\partial f}{\partial n} = \nabla f \cdot \mathbf{n} = \frac{\partial f}{\partial x_i} n_i \tag{2.6}$$

2.1.2.1 Exercise 2: Laplace operator

Determine the Laplace operator, i.e., the scalar product of the nabla vector with itself in Polar coordinates.

In order to express the *cross products* in the indicial notation, the third order tensor ϵ_{ijk}, known as the *permutation symbol*, must be introduced:

$$\epsilon_{ijk} = \begin{array}{cl} 1 & \text{if the values of } i,j,k \text{ are an even permutation of } 1,2,3 \\ & \text{(i.e. if they appear in sequence as in the arrangement 12312...)} \\ -1 & \text{if the values of } i,j,k \text{ are an odd permutation of } 1,2,3 \\ & \text{(i.e. if they appear in sequence as in the arrangement 32132...)} \\ 0 & \text{if the values of } i,j,k \text{ are not a permutation of } 1,2,3 \\ & \text{(i.e. if two or all three of the indices have the same value)} \end{array} \tag{2.7}$$

From this definition, the indicial notation of cross products is written by, e.g.

$$\mathbf{a} \times \mathbf{b} = \mathbf{c}, \ \epsilon_{ijk} a_j b_k = c_i \tag{2.8}$$

$$\nabla \times \mathbf{E} = \mathbf{D}, \ \epsilon_{ijk} \partial_j E_{kl} = D_{il} \tag{2.9}$$

$$\mathbf{D} \times \nabla = \mathbf{N}, \ D_{il} \partial_m \epsilon_{nlm} = N_{in} \tag{2.10}$$

Example: The *coss product* of of two vectors **a** and **b** may also be expanded as

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \hat{\mathbf{e}}_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\mathbf{e}}_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{\mathbf{e}}_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{aligned} \tag{2.11}$$

2.1.3 The Euclidian distance r and its derivatives

Considering two points $\mathbf{x}$ and ξ with its Cartesian coordinates (x_1, x_2, x_3) and (ξ_1, ξ_2, ξ_3), respectively, their Euclidian distance r is defined as

$$r = \mid \mathbf{x} - \xi \mid = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2} \tag{2.12}$$

Using the summation convention, r may be written in the indicial notation form as

$$r = \sqrt{\delta_{ij}(x_i - \xi_i)(x_j - \xi_j)} = \sqrt{(x_i - \xi_i)(x_i - \xi_i)} = [(x_i - \xi_i)(x_i - \xi_i)]^{1/2} \tag{2.13}$$

where δ_{ij} is the so-called Kronecker delta meaning

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \tag{2.14}$$

The first derivative of r with respect to x_j follows from (2.13) to be

$$\frac{\partial r}{\partial x_j} = r_{,j} = \frac{1}{2} [(x_i - \xi_i)(x_i - \xi_i)]^{-1/2} 2(x_j - \xi_j) = \frac{(x_j - \xi_j)}{r} \tag{2.15}$$

where, here, the comma is used to denote partial derivatives with respect to the coordinates of the point $\mathbf{x}$. The first derivative of r in special directions, e.g., in the direction of a normal vector n_i or a tangential vector t_i is simply

$$r_{,n} = r_{,i} n_i \quad \text{and} \quad r_{,t} = r_{,i} t_i \tag{2.16}$$

In R^1, this first derivatives of r may also be expressed as

$$r_{,1} = \frac{(x_1 - \xi_1)}{\mid x_1 - \xi_1 \mid} = \text{sign}(x_1 - \xi_1) = 2H(x_1 - \xi_1) - 1 \tag{2.17}$$

where $\text{sign}(x_1 - \xi_1)$ gives the sign of $(x_1 - \xi_1)$ and $H(x_1 - \xi_1)$ means the Heaviside function. Hence, as shown above , in R^1, the second derivative of r is

$$r_{,11} = 2\delta(x_1 - \xi_1) \tag{2.18}$$

This is different in R^2 and R^3. There, since $\partial(x_j - \xi_j)/\partial x_k = \delta_{jk}$, the second derivative is obtained to be ($j, k = 1, 2$ and $j, k = 1, 2, 3$ in R^2 and R^3, respectively)

$$r_{,jk} = \frac{\partial^2 r}{\partial x_j \partial x_k} = \frac{\delta_{jk}}{r} - \frac{(x_j - \xi_j)}{r^2} \frac{\partial r}{\partial x_k} = \frac{\delta_{jk} - r_{,j}\, r_{,k}}{r} \tag{2.19}$$

Examples: If the summation rule is applied, one obtains

$$\delta_{ii} = \begin{cases} 2 \text{ in } R^2 \\ 3 \text{ in } R^3 \end{cases}$$

and, in R^1, R^2, and in R^3

$$r_{,j}\, r_{,j} = 1$$

whereas

$$\begin{aligned} r_{,11} &= 2\delta(x_1 - \xi_1) \text{ in } R^1 \\ r_{,jj} &= r_{,11} + r_{,22} = \frac{1}{r} \text{ in } R^2 \\ r_{,jj} &= r_{,11} + r_{,22} + r_{,33} = \frac{2}{r} \text{ in } R^3 \end{aligned}$$

Remark: In general, i.e., when the normal vector n_i is defined at a curved boundary, it holds

$$r_{,in} = r_{,ij} n_j \neq r_{,ni} = (r_{,j} n_j)_{,i} = r_{,ji} n_j + r_{,j} n_{j,i}$$

Since with the curvature radius ρ of the boundary

$$n_{j,t} = \frac{1}{\rho} t_j \quad \text{and} \quad n_{j,n} = 0$$

one finds

$$\begin{aligned} n_{j,i} &= n_{j,t} t_i + n_{j,n} n_i \\ &= \frac{1}{\rho} t_j t_i \end{aligned}$$

it holds

$$r_{,ni} = (r_{,j} n_j)_{,i} = r_{,ji} n_j + r_{,j} n_{j,i} = r_{,ji} n_j + \frac{1}{\rho} t_j t_i$$

2.2 The Gauss theorems

The Gauss-Green theorem is a fundamental identity that relates a domain integral of a derivative of a tensorial function to an integral of that function around the boundary of that domain. The only requirement is that the integrand of the domain integral is a derivative, i.e., may be expressed as a product (inner, outer or cross) of the Nabla-vector ∇ with a tensorial function.

Let Ω be a finite domain in R^n bounded by a piecewise smooth orientable surface Γ with the outward normal vector $\mathbf{n}$ at a point on Γ. Dependent on the order of the tensorial function, following special theorems are known.

2.2.1 The gradient theorem

When F is a scalar function, the following identity holds

$$\int_\Omega \nabla F \, d\Omega = \oint_\Gamma \mathbf{n} F \, d\Gamma \tag{2.20}$$

where, obviously, the integrand of the surface integral is obtained by simply exchanging the Nabla-vector ∇ by the normal vector $\mathbf{n}$.

2.2.2 The divergence theorem

When $\mathbf{a}$ is a vector function and the Nabla-vector ∇ is multiplied via the inner product with this function, the divergence theorem holds

$$\int_\Omega \nabla \cdot \mathbf{a} \, d\Omega = \oint_\Gamma \mathbf{n} \cdot \mathbf{a} \, d\Gamma \quad \text{or} \quad \int_\Omega \partial_i a_i \, d\Omega = \oint_\Gamma n_i a_i \, d\Gamma = \oint_\Gamma a_n \, d\Gamma \tag{2.21}$$

which is also known as Gauss theorem. Similarly, when $\mathbf{S}$ is a dyadic function, one obtains

$$\int_\Omega \nabla \cdot \mathbf{S} \, d\Omega = \oint_\Gamma \mathbf{n} \cdot \mathbf{S} \, d\Gamma \quad \text{or} \quad \int_\Omega \partial_i S_{ij} \, d\Omega = \oint_\Gamma n_i S_{ij} \, d\Gamma = \oint_\Gamma t_j \, d\Gamma \tag{2.22}$$

2.2.3 Generalized Gauss theorems

Not often used, but nevertheless valid are generalizations where the cross product or the outer product is applied for multiplication:

$$\int_\Omega \nabla \times \mathbf{a} \, d\Omega = \oint_\Gamma \mathbf{n} \times \mathbf{a} \, d\Gamma \quad \text{or} \quad \int_\Omega \epsilon_{ijk} \partial_j a_k \, d\Omega = \oint_\Gamma \epsilon_{ijk} n_j a_k \, d\Gamma \tag{2.23}$$

$$\int_\Omega \nabla \mathbf{u} \, d\Omega = \oint_\Gamma \mathbf{n}\mathbf{u} \, d\Gamma \quad \text{or} \quad \int_\Omega \partial_i u_j \, d\Omega = \oint_\Gamma n_i u_j \, d\Gamma \tag{2.24}$$

Example 1.1

The realization of formula (2.20) in R^1 gives a well known result, when one notices that in this special case the domain Ω is simply an intervall $[a, b]$, the boundary Γ means here only the two points $x_1 = a$ and $x_1 = b$ and, hence, boundary integration is summation at these two points. Since, moreover, the outward normal vector at these points is $n_1(a) = -1$ and $n_1(b) = 1$,respectively, one obtains

$$\begin{aligned} \int_\Omega \nabla F \, d\Omega &= \oint_\Gamma \mathbf{n} F \, d\Gamma \\ \int_a^b \frac{d}{dx_1} F(x_1) dx_1 &= n_1(a)F(a) + n_1(b)F(b) = [F(x_1)]_{x_1=a}^{x_1=b} \end{aligned} \tag{2.25}$$

exactly as one knows it from basic calculus.

2.3 Integration by parts - Green's identities

By applying the above Gauss theorems, one can easily perform integrations by part. When the Laplace operator $\Delta = \nabla^2 = \nabla \cdot \nabla$ acts on the scalar function v and this result is multiplied with the scalar function u, one obtains in symbolic notation

$$\begin{aligned} \int_\Omega (\nabla \cdot \nabla v) \, u \, d\Omega &= \int_\Omega [\nabla \cdot ((\nabla v) u) - (\nabla v) \cdot (\nabla u)] \, d\Omega \\ &= \oint_\Gamma \mathbf{n} \cdot (\nabla v) \, u d\Gamma - \int_\Omega (\nabla v) \cdot (\nabla u) \, d\Omega \end{aligned} \tag{2.26}$$

or in indicial notation

$$\begin{aligned} \int_\Omega \frac{\partial^2 v}{\partial x_i \partial x_i} u \, d\Omega &= \int_\Omega \left[\frac{\partial}{\partial x_i} \left(\frac{\partial v}{\partial x_i} u \right) - \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i} \right] d\Omega \\ &= \oint_\Gamma n_i \frac{\partial v}{\partial x_i} u d\Gamma - \int_\Omega \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i} \, d\Omega \end{aligned} \tag{2.27}$$

Changing the sequence of these terms, this is the so-called Green's first identity

$$\int_\Omega [(\nabla \cdot \nabla v) \, u + (\nabla v) \cdot (\nabla u)] \, d\Omega = \oint_\Gamma \mathbf{n} \cdot (\nabla v) \, u d\Gamma \tag{2.28}$$

When the remaining domain integral in (2.26) is integrated by parts a second time

$$\begin{aligned} \int_\Omega (\nabla v) \cdot (\nabla u) \, d\Omega &= \int_\Omega [\nabla \cdot (v \, (\nabla u)) - v \, (\nabla \cdot \nabla u)] \, d\Omega \\ &= \oint_\Gamma \mathbf{n} \cdot (v \, (\nabla u)) \, d\Gamma - \int_\Omega v \, (\nabla \cdot \nabla u) \, d\Omega \end{aligned} \tag{2.29}$$

the final result is an additional boundary integral and a domain integral where all diffen-tiations are shifted from v to u:

$$\begin{aligned} \int_\Omega (\nabla \cdot \nabla v) \, u \, d\Omega &= \oint_\Gamma [\mathbf{n} \cdot (\nabla v) \, u - v \, \mathbf{n} \cdot (\nabla u)] \, d\Gamma + \int_\Omega v \, (\nabla \cdot \nabla u) \, d\Omega \\ \int_\Omega \frac{\partial^2 v}{\partial x_i \partial x_i} u \, d\Omega &= \oint_\Gamma \left[\frac{\partial v}{\partial n} u - v \frac{\partial u}{\partial n} \right] d\Gamma + \int_\Omega v \frac{\partial^2 u}{\partial x_i \partial x_i} \, d\Omega \end{aligned} \tag{2.30}$$

This is exactly the transformation which one needs for deriving an integral representation of the Laplace equation. In the form

$$\int_\Omega \left[(\nabla^2 v)\, u - v\, (\nabla^2 u)\right]\, d\Omega = \oint_\Gamma [\mathbf{n}\cdot(\nabla v)\, u - v\, \mathbf{n}\cdot(\nabla u)]\, d\Gamma \tag{2.31}$$

the relation (2.30) is known as Green's second identity.

The above demonstrated steps of integration by parts are naturally possible also for vectorial and tensorial states, e.g. with $\nabla u \triangleq \partial_i u_j = u_{j,i}$ and $\sigma \triangleq \sigma_{ij}$

$$\begin{aligned}
\int_\Omega (\nabla u)\cdot\cdot\sigma\, d\Omega &\triangleq \int_\Omega u_{j,i}\, \sigma_{ij}\, d\Omega \\
&= \int_\Omega \left[(u_j\, \sigma_{ij})_{,i} - u_j\, \sigma_{ij,i}\right]\, d\Omega \\
&= \oint_\Gamma u_j\, \sigma_{ij} n_i\, d\Gamma - \int_\Omega u_j\, \sigma_{ij,i}\, d\Omega \\
&= \oint_\Gamma (\mathbf{n}\cdot\sigma)\cdot \mathbf{u}\, d\Gamma - \int_\Omega (\nabla\cdot\sigma)\cdot\mathbf{u}\, d\Omega
\end{aligned} \tag{2.32}$$

Example 1.2

Again, the realization in R^1, here of formula (2.27), gives a well known result, when one notices that in this special case the domain Ω is simply an intervall $[a, b]$, the boundary Γ means here only the two points $x_1 = a$ and $x_1 = b$ and, hence, boundary integration is summation at these two points. Since, moreover, the outward normal vector at these points is $n_1(a) = -1$ and $n_1(b) = 1$, respectively, one obtains from (2.27) ($\frac{df}{dx} = f'$)

$$\begin{aligned}
\int_a^b f''(x_1) g(x_1) dx_1 &= [n_1(a) f'(a) g(a) + n_1(b) f'(b) g(b)] - \int_a^b f'(x_1) g'(x_1) dx_1 \\
&= [f'(x_1) g(x_1)]_{x_1=a}^{x_1=b} - \int_a^b f'(x_1) g'(x_1) dx_1
\end{aligned} \tag{2.33}$$

exactly what one knows from basic calculus.

2.3.0.1 Exercise 3: Integration by parts

Evaluate by integrations by parts the following integrals

a) in R^1 on the intervall $a < x < b$:

$$\int_a^b x^n \ln(x) dx$$

b) in R^2 on the circular domain $\Omega = \left\{(x_1, x_2) \mid r = \sqrt{x_1^2 + x_2^2} \leq R\right\}$:

$$\int_0^R r^n \ln(r) d\Omega$$

2.4 Fundamental solutions of differential equations

Let R^n denote the the Euclidian n-space and the ordered n-tuple $k = (k_1, k_2, ..., k_n)$ with non-negative integers $k_1, ..., k_n$ be a *multi-index* of dimension n. If we let $\mid k \mid= k_1 + ... + k_n$, then the k-th (partial) differential operator is defined by

$$D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} = \frac{\partial^{k_1 + ... + k_n}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}, \quad \mathbf{x} = (x_1, ..., x_n) \in R^n \tag{2.34}$$

such that if any component of k is zero, the partial derivative with respect to that variable is omitted. Moreover,

$$D^k u(\mathbf{x}) = \frac{\partial^{|k|} u(x_1, ..., x_n)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}, \quad D^0 u(\mathbf{x}) = u(\mathbf{x}) \tag{2.35}$$

$$\mathbf{D} = (D_1, D_2, ..., D_n) \text{ with } D_i = \partial_i = \frac{\partial}{\partial x_i}, \; i = 1, 2, ..., n. \tag{2.36}$$

An arbitrary linear differential operator L of order p in n independent variables $x_1, ..., x_n$ is denoted by

$$L \equiv L(D) = \sum_{|k| \leq p} a_k(\mathbf{x}) D^k \tag{2.37}$$

where the coefficients $a_k(\mathbf{x}) = a_{(k_1, k_2, ..., k_n)}(x_1, ..., x_n)$ are arbitrary functions.

2.4.1 Adjoint and self-adjoint operators

For the arbitrary linear differential operator L (2.37), the so-called *adjoint* operator L^* is formally defined by

$$L^* v = \sum_{|k| \leq p} (-1)^k D^k (a_k v). \tag{2.38}$$

If $a_k(\mathbf{x}) = a_k$ are constant, then $L^*(D) = L(-D)$. An operator is said to be *self-adjoint* if $L = L^*$.

Considering the *boundary value problem*

$$L(D) u(\mathbf{x}) = f(\mathbf{x}) \text{ in } \Omega \subset R^n, \tag{2.39}$$

$$B(u) = 0 \text{ on } \partial\Omega = \Gamma \tag{2.40}$$

where the Eq (2.40) represents linear boundary conditions, and introducing the *inner product* $\langle f_1, f_2\rangle$ of two functions f_1 and f_2 in the euclidean space $\Omega \subset R^n$ as

$$\langle f_1, f_2\rangle = \int\limits_{\Omega} f_1(\mathbf{x}) f_1(\mathbf{x}) d\Omega_{\mathbf{x}} \tag{2.41}$$

are more specific definition of L^* can be formulated:

Through integrations by part, i.e., by shifting all differentiations acting in Ω on the functions u to the functions w, one obtains

$$\langle Lu, w\rangle = \langle u, L^* w\rangle + \int\limits_{\Gamma} \{E(w)N(u) - N(u)E(w)\}\, d\Gamma \tag{2.42}$$

Here, again, the operator L^* is *adjoint* to L, and, if $L^* = L$, then L is said to be *self-adjoint.*

The formula (2.42) represents the *variational formulation* for the equation (2.39), where $E(u)$ generates *essential boundary conditions (*which must be enforced at some points so as to have a unique solution), while $N(u)$ generates non-essential boundary conditions, also called the *natural boundary conditions,* depending on the degree of derivatives that appear in the operators E and N that respectively define them.

Then, the *fundamental solution* $u^*(\mathbf{x}, \xi)$ is the solution of (2.39) for the special case when $f(\mathbf{x})$ is replace by $\delta(\mathbf{x}, \xi)$, the so-called *Dirac* δ-function (whose exact mathematical definition is only possible considering the theory of distributions, see, e.g., [4])

$$L(D)u^*(\mathbf{x}, \xi) = \delta(\mathbf{x}, \xi) \tag{2.43}$$

The function $u^*(\mathbf{x}, \xi)$ is unique only up to a function $w^*(\mathbf{x}, \xi)$ which is the solution of the homogeneous equation $L(D)w^* = 0$, i.e., the function $u^* + w^*$ is also a fundamental solution for the operator $L(D)$:

$$L(D)\left(u^*(\mathbf{x}, \xi) + w^*(\mathbf{x}, \xi)\right) = L(D)u^*(\mathbf{x}, \xi) + L(D)w^*(\mathbf{x}, \xi) = \delta(\mathbf{x}, \xi) \tag{2.44}$$

2.4.2 The Dirac δ-function

This symbolic function or distribution has the following basic properties:

$$\delta(\mathbf{x}, \xi) = 0 \quad for\ \mathbf{x} \neq \xi \tag{2.45}$$

where, in general, it is only a function of the distance between the two points $\mathbf{x}$ and ξ, i.e.,

$$\delta(\mathbf{x}, \xi) = \delta(\mathbf{x} - \xi) = \delta(x_1 - \xi_1)\delta(x_2 - \xi_2) \cdots \delta(x_n - \xi_n) \tag{2.46}$$

$$\int_{\Omega} \Phi(\mathbf{x})\delta(\mathbf{x}, \xi)\, d\Omega_{\mathbf{x}} = \Phi(\xi) \ \text{ for } \xi \in \Omega \tag{2.47}$$

for all sufficiently 'smooth', i.e., continous functions $\Phi(\mathbf{x})$. (2.47) is the *selection property* which means that the δ-function, when involved in an integration process with another function, selects the value of the other function at the point where the δ-function $\delta(\mathbf{x}-\xi)$ has a zero argument, i.e., at $\mathbf{x}=\xi$. For the special case $\Phi(\mathbf{x})\equiv 1$, (2.47) gives

$$\int_\Omega \delta(\mathbf{x},\xi)\; d\Omega_{\mathbf{x}} = \int_\Omega \delta(\mathbf{x}-\xi)\; d\Omega_{\mathbf{x}} = 1 \;\text{ for } \xi \in\Omega \tag{2.48}$$

Example 1.3

For a rectangular plane domain Ω, i.e., $x_1 \in [a_1, b_1], x_2 \in [a_2, b_2]$, the postulate (2.47) gives with (2.46)

$$\begin{aligned}\int_\Omega \delta(\mathbf{x}-\xi)\; d\Omega_{\mathbf{x}} &= \int_\Omega \delta(x_1-\xi_1)\delta(x_2-\xi_2)\; d\Omega_{\mathbf{x}} \\ &= \int_{a_1}^{b_1} \delta(x_1-\xi_1)dx_1 \int_{a_2}^{b_2} \delta(x_2-\xi_2)dx_2 = 1 \;\text{ for } \xi \in\Omega\end{aligned}$$

$$\text{or} \quad \int_{a_i}^{bi} \delta(x_i-\xi_i)dx_i = 1 \;\text{ for } \xi_i \in [a_i, b_i],\; i=1,2 \tag{2.49}$$

The expression for the δ-function becomes more complicated when one introduces curvilinear co-ordinates, i.e., the considered differential equations are formulated using curvilinear co-ordinates.

Example 1.4

The transformation from rectangular Cartesian x_1, x_2 to plane polar co-ordinates r, φ. The transformation is given by

$$x_1 = u(r,\varphi) = r\cos\varphi, \quad x_2 = v(r,\varphi) = r\sin\varphi$$

and the Jacobian J of the transformation is $J = r$ which yields

$$d\Omega_{\mathbf{x}} \triangleq r\, drd\varphi$$

Then, the integral statement (2.48), e.g., for a circular domain with radius R around $\xi = \mathbf{0}$, becomes due to the fact that the δ-function is only a function of the distance $r =\mid \mathbf{x}-\xi \mid$ and not of the angle φ

$$\begin{aligned}\int_\Omega \delta(\mathbf{x})\; d\Omega_{\mathbf{x}} &= \int_\Omega \delta(x_1)\delta(x_2)\; d\Omega_{\mathbf{x}} = \int_0^R \int_0^{2\pi} \delta(r,\varphi)\, r\, drd\varphi \\ &= \int_0^R \delta(r,\varphi)\, r\, dr \int_0^{2\pi} d\varphi = \int_0^R \delta(r,\varphi)\, r\, dr 2\pi = 1\end{aligned}$$

and postulates

$$\delta(r,\varphi) = \frac{\delta(r)}{2\pi r} \tag{2.50}$$

Hence, if a partial differential equation is formulated in polar co-ordinates, the adequate fundamental solution has to satisfy

$$L(D)u^*(r,\varphi) = \frac{\delta(r)}{2\pi r}$$

and, correspondingly, using in R^3 spherical co-ordinates r, φ and ϑ

$$L(D)u^*(r,\varphi,\vartheta) = \frac{\delta(r)}{4\pi r^2} \tag{2.51}$$

2.4.2.1 The δ-function in $\mathbf{R}^1$ and the Heaviside function

In R^1, it can be shown that the δ-function can be handled algebraically as if it were an ordinary function, but one must always interpret any equation involving $\delta(x)$ as follows: if the equation is multiplied throughout by an arbitrary continuous function $f(x)$, and integrated over an intervall $[a,b]$ by using the δ-function's selection property, i.e.

$$\begin{aligned} \int_a^b \delta(x-\xi)f(x)dx &= f(\xi) \quad \text{for } \xi \in [a,b] \\ &= 0 \quad \text{for } \xi \notin [a,b] \end{aligned} \tag{2.52}$$

then the resulting equation is correct and involves only ordinary functions.

For example

$$(x-\xi)\delta(x-\xi) = 0 \tag{2.53}$$

because for any arbitrary continuous function $f(x)$ one obtains with $g(x) = (x-\xi)f(x)$

$$\int_a^b \delta(x-\xi)g(x)dx = g(\xi) = 0 \quad \text{for } \xi \in [a,b]$$

Similarly, one can state

$$w(x)\delta(x-\xi) = 0 \quad \text{if} \quad w(x=\xi) = 0 \tag{2.54}$$

Moreover, the familiar techniques of integration, such as integration by parts and substitution, can be shown to apply to integrals involving δ-functions. As an example, consider the integral

$$I = \int_{-\infty}^{\infty} \delta(g(x))\ f(x)dx$$

where $f(x)$ is an arbitrary continuous function and $g(x)$ is a monotonic function of x which vanishes when $x = \xi$. Write $y = g(x)$ and it follows that $dy = g'(x)dx$. The integral then becomes

$$I = \int_{-\infty}^{\infty} \delta(g(x))\ f(x)dx = \int_{-\infty}^{\infty} \delta(y)\ \psi(y)dy = \psi(0) = \frac{f(\xi)}{|\ g'(\xi)\ |}$$

with $\psi(y) = f(x)/ \mid g'(x) \mid$, where the modulus sign is to ensure that the integration is always from $-\infty$ to ∞. Consequently, it follows if $g(x = \xi) = 0$ that

$$\delta(g(x)) = \frac{\delta(x-\xi)}{\mid g'(\xi) \mid} \tag{2.55}$$

As a special case of (2.55) one obtains

$$\int_{-\infty}^{\infty} \delta(ax - b)\ f(x)dx = \frac{f(\frac{b}{a})}{\mid a \mid} \tag{2.56}$$

And, again by integration by parts, one obtains

$$\int_a^b \frac{\partial\delta(x-\xi)}{\partial x}\ f(x)dx = -\int_a^b \delta(x-\xi)\ f'(x)dx = -f'(\xi) \ \text{ for } \xi \in [a,b] \tag{2.57}$$

Besides, in R^1, the Dirac function can be considered to be the derivative of the *Heaviside unit function* H defined as (see, e.g., [3], p. 147)

$$\begin{aligned} H(x-\xi) &= 1 \ \text{ for } x > \xi \\ &= 0 \ \text{ for } x < \xi \end{aligned} \tag{2.58}$$

To see this, we integrate by parts ($a < \xi < b$)

$$\begin{aligned} \int_a^b \frac{\partial H(x-\xi)}{\partial x} f(x)dx &= [H(x-\xi)f(x)]_{x=a}^{x=b} - \int_a^b H(x-\xi)f'(x)dx \\ &= f(b) - \int_\xi^b f'(x)dx \\ &= f(b) - (f(b) - f(\xi)) = f(\xi) \end{aligned}$$

which yields by comparison with (2.52)

$$\frac{\partial H(x-\xi)}{\partial x} = \delta(x-\xi) \tag{2.59}$$

2.4.2.2 The δ-function in R^n with $n \geq 2$

In Cartesian co-ordinates, the δ-function in a n-dimensional geometric space is the product of n one-dimensional δ-functions, e.g., in R^3

$$\delta(\mathbf{x} - \xi) = \delta(x_1 - \xi_1)\delta(x_2 - \xi_2)\delta(x_3 - \xi_3). \tag{2.60}$$

The expressions for the δ-functions become much more complicated when one introduces curvilinear coordinates. To find corresponding forms, we will, for simplicity, confine ourselves to two-dimensional space. Suppose that we transform from Cartesian co-ordinates x_1, x_2 to curvilinear co-ordinates ξ_1, ξ_2 by means of the relations

$$x_1 = u(\xi_1, \xi_2),\ x_2 = v(\xi_1, \xi_2) \tag{2.61}$$

where u and v are single-valued, continously differentable functions of their arguments. Supposing that under this transformation $\xi_1 = \beta_1$ and $\xi_2 = \beta_2$ correspond to $x_1 = \alpha_1$ and $x_2 = \alpha_2$, respectively.

If one changes the co-ordinates according to (2.61), the equation

$$\int\int \Phi(x_1, x_2)\delta(x_1 - \alpha_1)\delta(x_2 - \alpha_2)dx_1dx_2 = \Phi(\alpha_1, \alpha_2)$$

becomes

$$\int\int \Phi(u, v)\delta\left[u(\xi_1, \xi_2) - \alpha_1)\right]\delta\left[v(\xi_1, \xi_2) - \alpha_2)\right]|J|\, d\xi_1 d\xi_2 = \Phi(\alpha_1, \alpha_2)$$

where $J = \partial(u, v)/\partial(\xi_1, \xi_2)$ is the *Jacobian* of (2.61).

Correspondingly, we may write

$$\delta\left[u(\xi_1, \xi_2) - \alpha_1)\right]\delta\left[v(\xi_1, \xi_2) - \alpha_2)\right]|J| = \delta(\xi_1 - \beta_1)\delta(\xi_2 - \beta_2)$$

or, provided $J \neq 0$

$$\delta(x_1 - \alpha_1)\delta(x_2 - \alpha_2) = \frac{\delta(\xi_1 - \beta_1)\delta(\xi_2 - \beta_2)}{|J|} \tag{2.62}$$

Example: Transformation from rectangular Cartesian co-ordinates x, y to plane polar co-ordinates r, θ:

$$x = r\cos\theta,\ y = r\sin\theta$$

Since the Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

one obtains the relation

$$\delta(x - x_0)\delta(y - y_0) = \frac{\delta(r - r_0)\delta(\theta - \theta_0)}{r}$$

if $x_0 = r_0\cos\theta_0$ and $y_0 = r_0\sin\theta_0$.

For the event that $J = 0$ at some so-called *singular point*, the considered transformation is no longer one-to-one and, moreover, some co-ordinate , then called *ignorable co-ordinate*, is either many-valued or has no determinate value at such a singular point of the transformation. When, for example, the co-ordinate ξ_2 is ignorable, the Jacobian has to be integrated with respect to this co-ordinate (see, [3], p.219)

$$J_1 = \int |J| \, d\xi_2$$

and, consequently, in this case when $J = 0$ for $x_1 = \alpha_1$, we have the relation, e.g., in R^2

$$\delta(x_1 - \alpha_1)\delta(x_2 - \alpha_2) = \frac{\delta(\xi_1 - \beta_1)}{|J_1|}$$

Example (see also Example 1.4): In the case of the transformation $x = r\cos\theta, \; y = r\sin\theta$, the Jacobian $J = r$ vanishes at $x = 0, y = 0$ or $r = 0$ which means that θ may take on any value at this point, i.e., is ignorable. It follows that

$$J_1 = \int_0^{2\pi} r d\theta = 2\pi r$$

and, hence,

$$\delta(x)\delta(y) = \frac{\delta(r)}{2\pi r}$$

2.4.3 Green's functions of boundary value problems

As explained above, a boundary value problem is described by a differential equation

$$L(D)u(\mathbf{x}) = f(\mathbf{x}) \;\text{ in } \Omega \subset R^n, \tag{2.63}$$

and associated prescribed boundary conditions, e.g., on $\Gamma = \Gamma_1 \cup \Gamma_2$

$$E(u(\mathbf{x})) = \bar{u}(\mathbf{x}) \;\text{ for } \mathbf{x} \in \Gamma_1 \tag{2.64}$$

$$N(u(\mathbf{x})) = \bar{q}(\mathbf{x}) \;\text{ for } \mathbf{x} \in \Gamma_2 \tag{2.65}$$

Consequently, besides the unknown solution $u(\mathbf{x})$ in the interior Ω, there are also unknown boundary reactions $E(u(\mathbf{x})) = u(\mathbf{x})$ on Γ_2 and $N(u(\mathbf{x})) = q(\mathbf{x})$ on Γ_1 which are, in general, not zero along the respective boundary.

The so-called *Green's function* $G^*(\mathbf{x}, \xi)$ of such a boundary value problem is defined as being a special fundamental solution, i.e.,

$$L(D)G^*(\mathbf{x}, \xi) = \delta(\mathbf{x}, \xi) \tag{2.66}$$

which satisfies homogeneous conditions for those boundary states which are prescribed in the actual problem, i.e.,

$$E(G^*(\mathbf{x}, \xi)) = 0 \;\text{ for } \mathbf{x} \in \Gamma_1 \tag{2.67}$$

$$N(G^*(\mathbf{x}, \xi)) = 0 \;\text{ for } \mathbf{x} \in \Gamma_2 \tag{2.68}$$

When such Green's function of a boundary value problem is available - in general, it is difficult to construct it -the problem's solution can be formulated in an integral form which contains no unknown terms (see, section 3.1.3).

2.4.4 Ordinary differential equations with constant coefficients

The fundamental solution of an ordinary differential equation of n-th order with constant coefficients satisfies

$$L(D)u^*(x,\xi) = \frac{d^n u^*}{dx^n} + a_1 \frac{d^{n-1}u^*}{dx^{n-1}} + \cdots + a_{n-1}\frac{du^*}{dx} + a_n u^* = \delta(x,\xi) \tag{2.69}$$

and is given by

$$u^*(x,\xi) = H(x-\xi)w(x,\xi) \tag{2.70}$$

where $w(x,\xi) \in C^n(R^1)$ satisfies the homogeneous equation $L(D)w(x,\xi) = 0$ with the conditions

$$w(x=\xi) = \left.\frac{dw}{dx}\right|_{x=\xi} = \cdots = \left.\frac{d^{n-2}w}{dx^{n-2}}\right|_{x=\xi} = 0, \quad \left.\frac{d^{n-1}w}{dx^{n-1}}\right|_{x=\xi} = 1 \tag{2.71}$$

Since, in view of (2.59) and (2.54),

$$\frac{du^*(x,\xi)}{dx} = H(x-\xi)w'(x,\xi), ..., \frac{d^{n-1}u^*(x,\xi)}{dx^{n-1}} = H(x-\xi)w^{(n-1)}(x,\xi)$$

and

$$\frac{d^n u^*(x,\xi)}{dx^n} = \delta(x-\xi) + H(x-\xi)w^{(n)}(x,\xi),$$

one finds with $L(D)w(x,\xi) = 0$ that

$$\begin{aligned} L(D)u^*(x,\xi) &= \delta(x-\xi) \\ &\quad + H(x-\xi)\left[w^{(n)}(x,\xi) + a_1 w^{(n-1)}(x,\xi) + \cdots + a_n w(x,\xi)\right] \\ &= \delta(x-\xi) + H(x-\xi)L(D)w(x,\xi) \\ &= \delta(x-\xi) \end{aligned}$$

Example 1.5

From the above rule (2.70) with the conditions (2.71) follows that the fundamental solutions for the operators $L_1 = \frac{d}{dx} + a$, $L_2 = \frac{d^2}{dx^2} + a^2$, and $L_3 = \frac{d^2}{dx^2} - a^2$ are given by (see, [4], p.40)

$$u_1^*(x,\xi) = H(x-\xi)e^{-a(x-\xi)} \tag{2.72}$$

where, here for $n = 1$, $w_1(x,\xi) = e^{-a(x-\xi)}$ and satisfies the homogeneous differential equation $\frac{dw}{dx} + aw = 0$ and the condition $\left.\frac{d^0 w}{dx^0}\right|_{x=\xi} = w(x=\xi) = 1$, while for $n = 2$ the fundamental solutions are

$$u_2^*(x,\xi) = H(x-\xi)\frac{\sin a(x-\xi)}{a} \tag{2.73}$$

$$u_3^*(x,\xi) = H(x-\xi)\frac{\sinh a(x-\xi)}{a} \tag{2.74}$$

where the function $w_2(x,\xi) = \frac{1}{a}\sin a(x-\xi)$ and $w_3(x,\xi) = \frac{1}{a}\sinh a(x-\xi)$, respectively, satisfies again the related homogeneous differential equation and the conditions $w(x = \xi) = 0$ and $\frac{dw}{dx}\Big|_{x=\xi} = 1$.

When the constant a in the operator L_2 and L_3 tends to zero, one obtains the operator L_{bar} of the *bar equation* for constant stiffness EA (which also represents the potential equation in R^1)

$$L_{bar}u(x) = \frac{d^2u(x)}{dx^2} = \frac{p(x)}{EA} = f(x) \tag{2.75}$$

whose fundamental solution, i.e., the solution for $f(x) = \delta(x-\xi)$ is found by considering the limit of (2.73) for $a \to 0$:

$$u^*(x,\xi) = H(x-\xi)\lim_{a\to 0}\frac{\sin a(x-\xi)}{a} = (x-\xi)H(x-\xi) \tag{2.76}$$

But, as expressed in (2.44), this function $u^*(x,\xi)$ is unique only up to a function $w^*(x,\xi)$ which is the solution of the homogeneous equation $L_{bar}w^* = 0$, i.e., the function $u^* + w^*$ with $w^* = -\frac{1}{2}(x-\xi)$ is also a fundamental solution for the operator L_{Bar}

$$\begin{aligned} u^*_{bar}(x,\xi) &= (x-\xi)H(x-\xi) - \frac{1}{2}(x-\xi) = \frac{1}{2}(x-\xi)\left[2H(x-\xi)-1\right] \\ &= \frac{1}{2}(x-\xi)\text{sign}(x-\xi) = \frac{\mid x-\xi \mid}{2} = \frac{r}{2} \end{aligned} \tag{2.77}$$

where $\text{sign}(x-\xi) = 1$ for $x > \xi$ and $\text{sign}(x-\xi) = -1$ for $x < \xi$. This is obviously correct since $(x-\xi)\delta(x-\xi) = 0$ and

$$\begin{aligned} \frac{\partial}{\partial x}u^*_{bar}(x,\xi) &= \frac{1}{2}\frac{\partial r}{\partial x} = H(x-\xi) - \frac{1}{2} + \frac{1}{2}(x-\xi)\left[2\delta(x-\xi)\right] \\ &= H(x-\xi) - \frac{1}{2} = \frac{1}{2}\text{sign}(x-\xi) \end{aligned} \tag{2.78}$$

$$\frac{\partial^2}{\partial x^2}u^*(x,\xi) = \delta(x-\xi) \tag{2.79}$$

Correspondingly, a fundamental solution of the basic differential equation for an elastic beam with constant flexural rigidity EI

$$L_{beam}u(x) = \frac{d^4u(x)}{dx^4} = \frac{q(x)}{EI} = f(x) \tag{2.80}$$

can easily found by integrating the solution (2.77)of (2.75) twice:

$$u^*_{beam}(x,\xi) = \frac{r^3}{12} \tag{2.81}$$

2.4.5 Scalar partial differential equations with constant coefficients

The fundamental solution for the three-dimensional *Laplace equation*

$$-\Delta u^*(\mathbf{x},\xi) = \delta(\mathbf{x},\xi) \tag{2.82}$$

can be obtained directly as follows ([4], page 79):

Since the operator $\Delta = \nabla \cdot \nabla$ is invariant under a rotation of coordinate axes, we shall seek a solution that depends only on the distance $r =| \mathbf{x} - \xi |$. For $r > 0$, $u^*(\mathbf{x},\xi) = u^*(r)$ will satisfy the homogeneous equation $\Delta u^* = 0$, i.e., in spherical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u^*}{\partial r}\right) = 0, \tag{2.83}$$

which has a solution $u^* = \frac{A}{r} + B$. If one requires the solution u^* to vanish at infinity, then $B = 0$. In order to determine A, one has to take into account the magnitude of the source at $\mathbf{x} = \xi$. Integrating (2.82) over a small sphere Ω_ε of radius ε and center at $\mathbf{x} = \xi$, one obtains

$$-\int\limits_{\Omega_\varepsilon} \nabla \cdot \nabla u^*(\mathbf{x},\xi) d\Omega_\mathbf{x} = \int\limits_{\Omega_\varepsilon} \delta(\mathbf{x},\xi) d\Omega_\mathbf{x} = 1 \tag{2.84}$$

which, by using the divergence theorem (2.21), gives

$$-\int\limits_{\Gamma_\varepsilon} \mathbf{n} \cdot \nabla u^*(\mathbf{x},\xi) d\Gamma_\mathbf{x} = -\int\limits_{\Gamma_\varepsilon} \frac{\partial u^*(\mathbf{x},\xi)}{\partial n} d\Gamma_\mathbf{x} = -\int\limits_{\Gamma_\varepsilon} \frac{\partial u^*(\mathbf{x},\xi)}{\partial r} d\Gamma_\mathbf{x} = 1 \tag{2.85}$$

where $\Gamma_\varepsilon = \partial\Omega_\varepsilon$ is the surface of the sphere Ω_ε and $d\Gamma_\mathbf{x} = r^2 \sin\theta \, d\theta d\varphi$ with $0 \le \theta \le \pi$ and $-\pi \le \varphi \le \pi$. Now, substituting $u^* = A/r$, i.e., $\partial u^*/\partial r = -A/r^2$ in (2.85) yields

$$\int\limits_{-\pi}^{\pi}\int\limits_{0}^{\pi} \frac{A}{r^2} r^2 \sin\theta \, d\theta d\varphi = A \int\limits_{-\pi}^{\pi} d\varphi \int\limits_{0}^{\pi} \sin\theta \, d\theta = A2\pi \left[-\cos\theta\right]_0^\pi = 4\pi A = 1$$

i.e., $A = 1/(4\pi)$. Hence, the fundamental solution for the three-dimensional Laplace equation is

$$u^*(\mathbf{x},\xi) = u^*(r) = \frac{1}{4\pi r} \tag{2.86}$$

Correspondingly, the homogeneous two-dimensional Laplace equation in polar coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u^*}{\partial r}\right) = 0, \tag{2.87}$$

has for $r > 0$ the solution $u^* = C \ln(r/a) + D$. Arbitrarily setting $D = 0$ and applying the same steps as above in (2.84) and (2.85), determines $C = -1/(2\pi)$, and hence the fundamental solution of the two-dimensional Laplace equation is

$$u^*(\mathbf{x},\xi) = u^*(r) = \frac{-1}{2\pi} \ln(\frac{r}{a}) = \frac{1}{2\pi} \ln(\frac{a}{r}) \tag{2.88}$$

where $a > 0$ is an arbitrary real constant making the ratio r/a dimensionless.

Similarly (see, e.g., [5] or [4]), one can determine the fundamental solutions for the Helmholtz equations in $R^n, n = 1, 2, 3$

$$(\Delta + k^2)u^*(\mathbf{x}, \xi) = \delta(\mathbf{x}, \xi) \tag{2.89}$$

where $k \neq 0$, real, is the so-called wave number. One gets

$$\text{for } n = 3: \; u^*(\mathbf{x}, \xi) = u^*(r) = -\frac{1}{4\pi r}e^{-ikr} \tag{2.90}$$

$$\text{for } n = 2: \; u^*(\mathbf{x}, \xi) = u^*(r) = \frac{i}{4}H_0^{(2)}(kr) \tag{2.91}$$

$$\text{for } n = 1: \; u^*(x, \xi) = u^*(r) = \frac{1}{2k}\sin(kr) \tag{2.92}$$

where $H_0^{(2)}(kr)$ is a Hankel function. With $h = -ik$ or $k = ih$, the operator (2.89) changes to

$$(\Delta - h^2)u^*(\mathbf{x}, \xi) = \delta(\mathbf{x}, \xi) \tag{2.93}$$

and the fundamental solutions become

$$\text{for } n = 3: \; u^*(\mathbf{x}, \xi) = u^*(r) = -\frac{1}{4\pi r}e^{ihr} \tag{2.94}$$

$$\text{for } n = 2: \; u^*(\mathbf{x}, \xi) = u^*(r) = -\frac{1}{2\pi}K_0(hr) \tag{2.95}$$

$$\text{for } n = 1: \; u^*(x, \xi) = u^*(r) = -\frac{1}{2h}\sinh(hr) \tag{2.96}$$

where $K_0(hr)$ is a modified Besselfunction. Note that here $\lim_{r\to\infty} u^*(r) = \infty$, i.e., these fundamental solutions represent wave forms that diverge to infinity.

Remark 1: The R^1 fundamental solutions given in (2.92) and (2.73) for the operator (2.89) as well as in (2.96) and (2.74) for the operator (2.93) are different, but, as one can easily check, both versions are correct, i.e., give the typical filtering effect of the Dirac δ-function. For example, gives the fundamental solution (2.92) $u^*(x, \xi) = u^*(r) = \frac{1}{2k}\sin(kr)$ of the equation (2.89)(note that $r_{,x} = \partial r/\partial x = 2H(x - \xi) - 1$ and $(r_{,x})^2 = 1$)

$$\int_a^b (\Delta + k^2)u^*(x, \xi)dx = \int_a^b \cos(kr)\delta(x, \xi)dx = 1 \;\text{ for } \xi \in [a, b]$$

while the fundamental solution (2.74) $u_2^*(x, \xi) = H(x - \xi)\frac{\sin k(x-\xi)}{k}$ for the same equation results in

$$\int_a^b (\Delta + k^2)u_2^*(x, \xi)dx = \int_a^b \cos(k(x - \xi))\delta(x, \xi)dx = 1 \;\text{ for } \xi \in [a, b]$$

The essential difference of both versions is that the forms (2.73) and (2.73) contain the Heaviside function $H(x-\xi)$ and, therefore, are cut off (zero) for $x<\xi$, while the other versions (2.92) and (2.96) are due to their dependence on $r=|x-\xi|$ symmetric to $x=\xi$.

Remark 2: In R^1, it is also possible to use instead of the divergent form (2.96)

$$u^*(x,\xi)=u^*(r)=-\frac{1}{2h}e^{-hr} \tag{2.97}$$

This form is convergent, i.e., $\lim_{r\to\infty}u^*(r)=0$, and has, as easily can be checked, the essential behaviour of a fundamental solution:

$$\int_a^b(\Delta-h^2)u^*(x,\xi)dx=\int_a^b e^{-hr}\delta(x,\xi)dx=1 \text{ for } \xi\in[a,b] \tag{2.98}$$

Remark 3: When taking into account that $u^*(r)=1/4\pi r$ is the fundamental solution of the Laplace operator in R^3, i.e.,

$$\Delta(\frac{1}{r})=\nabla^2(\frac{1}{r})=-4\pi\delta(\mathbf{x},\xi)$$

the check that $u^*=-\frac{1}{4\pi r}e^{-ikr}$ is really the fundamental solution of the Helmholtz equation (2.89) in R^3 is straight-forward by applying the Leibniz formula $\nabla^2(af)=f\nabla^2(a)+2\nabla(a)\cdot\nabla(f)+a\nabla^2(f)$, i.e. here,

$$(\Delta+k^2)u^*=-\frac{1}{4\pi}\left\{e^{-ikr}\nabla^2(\frac{1}{r})+2\nabla(\frac{1}{r})\cdot\nabla(e^{-ikr})+\frac{1}{r}\nabla^2(e^{-ikr})\right\}+k^2u^*$$

Since differentiations give ($\frac{\partial}{\partial x_i}=_{,i}$)

$$\begin{aligned}
\nabla(\frac{1}{r}) &= \frac{\partial}{\partial x_i}(\frac{1}{r})=-\frac{1}{r^2}r_{,i}=-\frac{(x_i-\xi_i)}{r^3}\\
\nabla(e^{-ikr}) &= \frac{\partial}{\partial x_i}(e^{-ikr})=-ikr_{,i}e^{-ikr}=-ik\frac{(x_i-\xi_i)}{r}e^{-ikr}\\
\nabla^2(e^{-ikr}) &= \frac{\partial}{\partial x_i}(-ik\frac{(x_i-\xi_i)}{r}e^{-ikr})=-(\frac{2ik}{r}+k^2)e^{-ikr}
\end{aligned}$$

one obtains finally

$$\begin{aligned}
(\Delta+k^2)u^* &= -\frac{1}{4\pi}\left\{\begin{array}{c}-e^{-ikr}4\pi\delta(\mathbf{x},\xi)+2\frac{(x_i-\xi_i)}{r^3}ik\frac{(x_i-\xi_i)}{r}e^{-ikr}\\ -\frac{1}{r}(\frac{2ik}{r}+k^2)e^{-ikr})\end{array}\right\}-\frac{k^2}{4\pi r}e^{-ikr}\\
&= e^{-ikr}\delta(\mathbf{x},\xi)=\delta(\mathbf{x},\xi)
\end{aligned}$$

Certainly, one should know that

$$\begin{aligned}
\frac{\partial^2}{\partial x_i\partial x_j}(e^{-ikr}) &= \frac{\partial}{\partial x_j}(-ikr_{,i}e^{-ikr})=-ik\left(r_{,ij}e^{-ikr}+r_{,i}(-ikr_{,j})e^{-ikr}\right)\\
&= -ik\left(r_{,ij}-ikr_{,i}r_{,j})\right)e^{-ikr}
\end{aligned}$$

with

$$r_{,ij}=\frac{\partial}{\partial x_j}\left(\frac{(x_i-\xi_i)}{r}\right)=\frac{\delta_{ij}}{r}-\frac{(x_i-\xi_i)r_{,j}}{r^2}=\frac{1}{r}\left(\delta_{ij}-r_{,i}r_{,j}\right)$$

2.5 Singular integrals

Singular integrals are those whose integrands reach an infinite value at some points on the integration domain Ω. They are, in general, defined by eliminating a small space (initially arbitrary) including the singular point, and obtaining the limit when this small space tends to disappear

$$\int_{\Omega} f(x,\xi) d\Omega_x = \lim_{\varepsilon\to 0} \int_{\Omega-\Omega_\varepsilon} f(x,\xi) d\Omega_x \text{ with } \xi \in \Omega_\varepsilon \tag{2.99}$$

Ω_ε can be a ball of radius ε in $3D$, a circle of radius ε in $2D$, and in $1D$, i.e. on a line a segment of dimension ε at each side of the point where the singularity is located.

2.5.1 Weak singularities - improper integrals

If the limit of (2.99) exists independently of how ε tends to zero, it is said that this integral exists as *improper* and the singularity is said to be weak.

In R^1, a representative example is the integral

$$\int_a^b \ln \mid x-\xi \mid dx, \quad a<\xi<b \tag{2.100}$$

which can evaluated as

$$\begin{aligned}
\int_a^b \ln|x-\xi|\, dx &= \lim_{\varepsilon_1\to 0,\varepsilon_2\to 0}\left(\int_a^{\xi-\varepsilon_1} \ln(\xi-x)dx + \int_{\xi+\varepsilon_2}^b \ln(x-\xi)dx\right)\\
&= \lim_{\varepsilon_1\to 0}\left[-(\xi-x)[\ln(\xi-x)-1]\right]_a^{\xi-\varepsilon_1} + \lim_{\varepsilon_2\to 0}\left[(x-\xi)[\ln(x-\xi)-1]\right]_{\xi+\varepsilon_2}^b\\
&= \lim_{\varepsilon_1\to 0}\left(-\varepsilon_1[\ln(\varepsilon_1)-1]\right) + (\xi-a)[\ln(\xi-a)-1]\\
&\quad +(b-\xi)[\ln(b-\xi)-1] - \lim_{\varepsilon_2\to 0}\left(\varepsilon_2[\ln(\varepsilon_2)-1]\right)\\
&= (a-b)+(\xi-a)\ln(\xi-a)+(b-\xi)\ln(b-\xi)
\end{aligned} \tag{2.101}$$

obviously exists, since from the rule of Bernoulli-de l'Hospital

$$\lim_{\varepsilon\to 0}(\varepsilon\ln(\varepsilon)) = \lim_{\varepsilon\to 0}\left(\frac{\ln(\varepsilon)}{\varepsilon^{-1}}\right) = \lim_{\varepsilon\to 0}\left(\frac{\varepsilon^{-1}}{-\varepsilon^{-2}}\right) = \lim_{\varepsilon\to 0}(-\varepsilon) = 0 \tag{2.102}$$

follows that all ε-terms disappear independently from each other.

Physically, this fact implies that the area under the function at any side if the singular point has a finite value.

Remark: The following integrals also exist as improper integrals

$$I = \int_a^b \frac{dx}{|x-\xi|^k} \quad \text{for } 0<k<1$$

2.5.2 The Cauchy Principal Value of strongly singular integrals

When one evaluates the integral

$$\int_a^b \frac{dx}{x-\xi}, \quad a < \xi < b \tag{2.103}$$

as an improper integral

$$\int_a^b \frac{dx}{x-\xi} = \lim_{\varepsilon_1 \to 0, \varepsilon_2 \to 0} \left(-\int_a^{\xi-\varepsilon_1} \frac{dx}{\xi - x} + \int_{\xi+\varepsilon_2}^b \frac{dx}{x-\xi} \right) = \ln \frac{b-\xi}{\xi - a} + \lim_{\varepsilon_1 \to 0, \varepsilon_2 \to 0} \ln \frac{\varepsilon_1}{\varepsilon_2} \tag{2.104}$$

the limit of the last expression obviously depends on the way in which ε_1 and ε_2 tend to zero. Hence, the improper integral does not exist. This integral is called a *strongly singular integral*. However, this integral can be assigned a meaning if we assume that there is some relationship between ε_1 and ε_2, e.g., if the deleted intervall is symmetric with respect to the point ξ, i.e., $\varepsilon_1 = \varepsilon_2 = \varepsilon$. Then, one obtains

$$\int_a^b \frac{dx}{x-\xi} = \lim_{\varepsilon \to 0} \left(-\int_a^{\xi-\varepsilon} \frac{dx}{\xi - x} + \int_{\xi+\varepsilon}^b \frac{dx}{x-\xi} \right) = \ln \frac{b-\xi}{\xi - a} \tag{2.105}$$

the so-called *Cauchy principal value* (CPV) of a singular integral.

Remark: Evaluating weakly singular integrals as Cauchy principal value gives obviously the same result.

Now, consider the more general integral

$$\int_a^b \frac{\varphi(x)}{x-\xi} dx, \quad a < \xi < b \tag{2.106}$$

where $\varphi(x), x \in [a,b]$ is a function satisfying the *Hölder condition*, i.e., for any two points t_1 and t_2 on a smooth curve L and for positive constants A and λ with $0 < \lambda \leq 1$ holds

$$\mid \varphi(t_2) - \varphi(t_1) \mid < A \mid t_2 - t_1 \mid^\lambda \tag{2.107}$$

which means that $\varphi(t)$ is differentiable and has a bounded derivative. The Hölder condition is sometimes referred to as an intermediate situation between continuity and derivability, establishing in fact a division in the set of continuous non-derivable functions (see, Paris and Canas [6], p. 14)

Let us understand this integral (2.106) in the sense of the Cauchy principal value (CPV) as

$$\int_a^b \frac{\varphi(x)}{x-\xi} dx = \lim_{\varepsilon \to 0} \left(\int_a^{\xi-\varepsilon} \frac{\varphi(x)}{x-\xi} dx + \int_{\xi+\varepsilon}^b \frac{\varphi(x)}{x-\xi} dx \right) \tag{2.108}$$

and take the identity

$$\int_a^b \frac{\varphi(x)}{x-\xi}dx = \int_a^b \frac{\varphi(x)-\varphi(\xi)}{x-\xi}dx + \varphi(\xi)\int_a^b \frac{dx}{x-\xi} \tag{2.109}$$

Now, one can see that the first integral on the right-hand side of (2.109) is convergent as an improper integral, because it follows from the Hölder condition that

$$\mid \frac{\varphi(x)-\varphi(\xi)}{x-\xi} \mid < \frac{A}{\mid x-\xi \mid^{1-\lambda}},\ 0<\lambda\leq 1 \tag{2.110}$$

and the second integral coincides with (2.105). Thus, if $\varphi(x)$ satisfies the Hölder condition, the singular integral (2.106) exists in the sense of the Cauchy principle value (CVP)and is equal to

$$\int_a^b \frac{\varphi(x)}{x-\xi}dx = \int_a^b \frac{\varphi(x)-\varphi(\xi)}{x-\xi}dx + \varphi(\xi)\ln\frac{b-\xi}{\xi-a} \tag{2.111}$$

Remark: The following integral does not exist, neither as improper integral nor as Cauchy Principal Value:

$$I = \int_a^b \frac{dx}{|x-\xi|}$$

Example: Determine the CVP of the following integral ($a<\xi<b$):

$$\begin{aligned}
\int_a^b \frac{dx}{\sin(x-\xi)} &= \int_a^{\xi-\varepsilon} \frac{dx}{-\sin(\xi-x)} + \int_{\xi+\varepsilon}^b \frac{dx}{\sin(x-\xi)} \\
&= \int_{y=\xi-a}^{\varepsilon} \frac{dy}{\sin(y)} + \int_{\varepsilon}^{b-\xi} \frac{dy}{\sin(y)} \\
&= \left[\ln\left(\tan(\frac{y}{2})\right)\right]_{\xi-a}^{\varepsilon} + \left[\ln\left(\tan(\frac{y}{2})\right)\right]_{\varepsilon}^{b-\xi} \\
&= \ln\left(\frac{\tan(\frac{b-\xi}{2})}{\tan(\frac{\xi-a}{2})}\right)
\end{aligned}$$

2.5.3 Cauchy Principal Value integrals in boundary integral equations

In boundary integral equations, the integrands contain often singularities of inverse powers of $r = \mid \mathbf{x}-\xi \mid$ multiplied by certain bounded functions, i.e., have the general structure

$$I = \int_\Gamma \frac{\varphi(\mathbf{x},\xi)}{\mid \mathbf{x}-\xi \mid^k} u(\mathbf{x}) d\Gamma_{\mathbf{x}}$$

where the so-called *characteristic* $\varphi(\mathbf{x},\xi)$ does not include any singularity, the singular *kernel* $\varphi(\mathbf{x},\xi)/r^k$ results from the fundamental solution while the so-called *density* $u(\mathbf{x})$ usually represents boundary values of the considered problem.

2.5.3.1 Conditions for the existence of a CPV of a singular curvilinear line integral

Let Γ be a smooth contour, $\mathbf{x}$ and ξ be coordinates of its points, and $\mathbf{x}_a$ and $\mathbf{x}_b$ be the endpoints of Γ. Consider the singular curvilinear integral where the contour points are expressed in terms of a parameter, e.g., the arc length s, so that $\mathbf{x}=\mathbf{x}(s)$ and s_a, s_b are the values of the parameter s corresponding to the endpoints of Γ:

$$\int\limits_{\Gamma}\frac{\varphi(\mathbf{x},\xi)}{|\,\mathbf{x}-\xi\,|}u(\mathbf{x})d\Gamma_{\mathbf{x}}=\int\limits_{s_a}^{s_b}\frac{\varphi(\mathbf{x}(s),\xi)}{|\,\mathbf{x}(s)-\xi\,|}u(\mathbf{x}(s))\frac{d\Gamma_{\mathbf{x}}}{ds}ds=\int\limits_{s_a}^{s_b}\frac{\varphi(\mathbf{x}(s),\xi)}{|\,\mathbf{x}(s)-\xi\,|}u(\mathbf{x}(s))J(s)ds \tag{2.112}$$

where $J(s)$ is a Jacobian associated with the change of variable from $\mathbf{x}=(x_1,x_2)$ to s

$$J(s)=\frac{d\Gamma_{\mathbf{x}}}{ds}=\sqrt{\left(\frac{dx_1}{ds}\right)^2+\left(\frac{dx_2}{ds}\right)^2} \tag{2.113}$$

Let us take a circle of some radius ε centered at the point ξ on the contour, $\xi_+=\xi+\mathbf{x}_\varepsilon$ and $\xi_-=\xi-\mathbf{x}_\varepsilon$ be the points of intersection of this circle with the curve, and assume that the radius is so small that the circle has no other points of intersection with Γ. Let Γ_ε be the part of the contour Γ cut out by the circle and consider the integral over the remaining arc. Then, its limit for $\varepsilon\to 0$ is the principal value of the singular integral (2.112).

$$\begin{aligned}\int\limits_{\Gamma}\frac{\varphi(\mathbf{x},\xi)}{|\,\mathbf{x}-\xi\,|}u(\mathbf{x})d\Gamma_{\mathbf{x}} &= \lim_{\varepsilon\to 0}\int\limits_{\Gamma-\Gamma_\varepsilon}\frac{\varphi(\mathbf{x},\xi)}{|\,\mathbf{x}-\xi\,|}u(\mathbf{x})d\Gamma_{\mathbf{x}}\\ &= \int\limits_{\Gamma}\frac{\varphi(\mathbf{x},\xi)}{|\,\mathbf{x}-\xi\,|}\left[u(\mathbf{x})-u(\xi)\right]d\Gamma_{\mathbf{x}}+u(\xi)\int\limits_{\Gamma}\frac{\varphi(\mathbf{x},\xi)}{|\,\mathbf{x}-\xi\,|}d\Gamma_{\mathbf{x}}\end{aligned} \tag{2.114}$$

where, if $u(\mathbf{x})$ satisfies the Hölder condition for points $\mathbf{x}$ being placed in the neighbourhood of ξ, the first integral exists. The second integral is, e.g., along a straight line contour of length $\Delta s=s_b-s_a$ where with $\mathbf{x}=\mathbf{x}(s)$ and $\xi=\mathbf{x}(\bar{s})$ the distance $r=|\,\mathbf{x}-\xi\,|$ is

$r =| s - \bar{s} |$ and $J(s) = 1$

$$\begin{aligned}
\int\limits_{\mathbf{x}_a}^{\mathbf{x}_b} \frac{\varphi(\mathbf{x}, \xi)}{| \mathbf{x} - \xi |} d\Gamma_{\mathbf{x}} &= \lim_{\varepsilon \to 0} \left[\int\limits_{s_a}^{\bar{s}-\varepsilon} \frac{\varphi(\mathbf{x}(s), \mathbf{x}(\bar{s}))}{|s - \bar{s} |} ds + \int\limits_{\bar{s}+\varepsilon}^{s_b} \frac{\varphi(\mathbf{x}(s), \mathbf{x}(\bar{s}))}{|s - \bar{s} |} ds \right] \qquad (2.115)\\
&= \lim_{\varepsilon \to 0} \left[\int\limits_{s_a}^{\bar{s}-\varepsilon} \frac{\varphi(\mathbf{x}(s), \mathbf{x}(\bar{s}))}{\bar{s} - s} ds + \int\limits_{\bar{s}+\varepsilon}^{s_b} \frac{\varphi(\mathbf{x}(s), \mathbf{x}(\bar{s}))}{s - \bar{s}} ds \right]\\
&= \lim_{\varepsilon \to 0} \left[\int\limits_{\bar{s}-\varepsilon}^{s_a} \frac{\varphi(\mathbf{x}(s), \mathbf{x}(\bar{s}))}{s - \bar{s}} ds + \int\limits_{\bar{s}+\varepsilon}^{s_b} \frac{\varphi(\mathbf{x}(s), \mathbf{x}(\bar{s}))}{s - \bar{s}} ds \right]\\
&= I_1 + I_2
\end{aligned}$$

Remark: In many discretizations, the boundary line of a domain is approximated by a polygon, i.e., by elementwise straight lines.

Evaluating I_1 by integration by parts gives

$$\begin{aligned}
I_1 &= \lim_{\varepsilon \to 0} \int\limits_{\bar{s}-\varepsilon}^{s_a} \frac{\varphi(\mathbf{x}(s), \mathbf{x}(\bar{s}))}{s - \bar{s}} ds\\
&= \lim_{\varepsilon \to 0} \left[\ln(|s - \bar{s} |) \varphi(\mathbf{x}(s), \mathbf{x}(\bar{s}))|_{\bar{s}-\varepsilon}^{s_a} - \int\limits_{\bar{s}-\varepsilon}^{s_a} \ln(|s - \bar{s} |) \frac{d\varphi(\mathbf{x}(s), \mathbf{x}(\bar{s}))}{ds} ds \right]\\
&= \lim_{\varepsilon \to 0} [\ln(|s_a - \bar{s} |) \varphi(\mathbf{x}(s_a), \mathbf{x}(\bar{s})) - \ln(\varepsilon) \varphi(\mathbf{x}(\bar{s} - \varepsilon), \mathbf{x}(\bar{s})) - I_{12}]
\end{aligned}$$

The integral I_{12} and the corresponding integral from I_2 exists if $d\varphi/dx$ takes finite values along the integration zone outside $r = 0$ while the first evaluated terms of I_1 and the corresponding ones from I_2 lead to

$$\lim_{\varepsilon \to 0} \left[\begin{array}{c} \ln(|s_a - \bar{s} |) \varphi(\mathbf{x}(s_a), \mathbf{x}(\bar{s})) - \ln(\varepsilon) \varphi(\mathbf{x}(\bar{s} - \varepsilon), \mathbf{x}(\bar{s})) \\ + \ln(|s_b - \bar{s} |) \varphi(\mathbf{x}(s_b), \mathbf{x}(\bar{s})) - \ln(\varepsilon) \varphi(\mathbf{x}(\bar{s} + \varepsilon), \mathbf{x}(\bar{s})) \end{array} \right]$$

Hence, if

$$\lim_{\varepsilon \to 0} \{\ln(\varepsilon) [\varphi(\mathbf{x}(\bar{s} - \varepsilon), \mathbf{x}(\bar{s})) + \varphi(\mathbf{x}(\bar{s} + \varepsilon), \mathbf{x}(\bar{s}))]\} = 0$$

and $d\varphi/dx$ takes finite values along the integration zone outside $r = 0$, the integral $I_1 + I_2$ would have a finite value:

$$\begin{aligned}
\lim_{\varepsilon \to 0} \left[\int\limits_{s_a}^{\bar{s}-\varepsilon} \frac{\varphi(\mathbf{x}(s), \mathbf{x}(\bar{s}))}{|s - \bar{s} |} ds + \int\limits_{\bar{s}+\varepsilon}^{s_b} \frac{\varphi(\mathbf{x}(s), \mathbf{x}(\bar{s}))}{|s - \bar{s} |} ds \right] &= \ln(|s_a - \bar{s} |) \varphi(\mathbf{x}(s_a), \mathbf{x}(\bar{s}))\\
&\quad + \ln(| s_b - \bar{s} |) \varphi(\mathbf{x}(s_b), \mathbf{x}(\bar{s}))\\
&\quad - \left[\int\limits_{\bar{s}-\varepsilon}^{s_a} + \int\limits_{\bar{s}+\varepsilon}^{s_b} \right] \ln(| s - \bar{s} |) \frac{d\varphi(\mathbf{x}(s), \mathbf{x}(\bar{s}))}{ds} ds \qquad (2.116)
\end{aligned}$$

This condition is satisfied by a function φ if in the neighbourhood of $\xi = \mathbf{x}(\bar{s})$ it holds that

$$|\varphi(\mathbf{x}(\bar{s}-\varepsilon),\mathbf{x}(\bar{s})) + \varphi(\mathbf{x}(\bar{s}+\varepsilon),\mathbf{x}(\bar{s}))| \leq A\varepsilon^{\alpha};\ A > 0;\ 0 < \alpha \leq 1;\ \varepsilon > 0 \tag{2.117}$$

2.5.3.2 Conditions for the existence of a CPV of a singular surface integral

For a study of singular integrals in R^2 where singularities of order two $(1/r^2)$ can have a Cauchy Principal Value, let us consider the integral on a domain Ω_2 in R^2, here, in general, a surface domain of three-dimensional areas

$$I(\xi) = \int\limits_{\Omega_2} \frac{\varphi(\xi,\theta)}{r^2} u(\mathbf{x}) d\Omega_{\mathbf{x}} \tag{2.118}$$

where $r = \mid \mathbf{x} - \xi \mid$ and θ represents the angle formed by r with respect to the coordinate axes. This integral can be divided in the following manner:

$$\begin{aligned}\int\limits_{\Omega_2} \frac{\varphi(\xi,\theta)}{r^2} u(\mathbf{x}) d\Omega_{\mathbf{x}} \quad &= \int\limits_{\Omega_2 \cap (r \geq \rho)} \frac{\varphi(\xi,\theta)}{r^2} u(\mathbf{x}) d\Omega_{\mathbf{x}} + \\ & \int\limits_{r<\rho} \frac{\varphi(\xi,\theta)}{r^2} \left[u(\mathbf{x}) - u(\xi)\right] d\Omega_{\mathbf{x}} + u(\xi) \int\limits_{r<\rho} \frac{\varphi(\xi,\theta)}{r^2} d\Omega_{\mathbf{x}}\end{aligned} \tag{2.119}$$

where ρ is a fixed distance sufficiently small to guarantee that the points $\mathbf{x} : |\mathbf{x} - \xi| < \rho$ belong to Ω_2.

The first integral of (2.119) is defined and the second exists if the Hölder condition is satisfied:

$$|u(\mathbf{x}) - u(\xi)| \leq A r^{\alpha}(\mathbf{x},\xi);\ A > 0;\ 0 < \alpha \leq 1. \tag{2.120}$$

The third integral of (2.119) is specified as

$$\int\limits_{r<\rho} \frac{\varphi(\xi,\theta)}{r^2} d\Omega_{\mathbf{x}} = \lim_{\varepsilon \to 0} \int\limits_{\varepsilon<r<\rho} \frac{\varphi(\xi,\theta)}{r^2} d\Omega_{\mathbf{x}} \tag{2.121}$$

and polar coordinates, i.e., $d\Omega_{\mathbf{x}} = r d\theta dr$ are introduced which yields

$$\lim_{\varepsilon \to 0} \int\limits_{\varepsilon<r<\rho} \frac{\varphi(\xi,\theta)}{r^2} r d\theta dr = \lim_{\varepsilon \to 0} \int\limits_{\varepsilon}^{\rho} \frac{1}{r} dr \int\limits_{0}^{2\pi} \varphi(\xi,\theta) d\theta = \lim_{\varepsilon \to 0} \ln(\frac{\rho}{\varepsilon}) \int\limits_{0}^{2\pi} \varphi(\xi,\theta) d\theta$$

The condition for this integral to exist is then:

$$\int\limits_{0}^{2\pi} \varphi(\xi,\theta) d\theta = 0 \tag{2.122}$$

Hence, if the integral of the characteristic $\varphi(\xi,\theta)$ on the surface around the pole $\mathbf{x} = \xi$ is zero and the density $u(\mathbf{x})$ satisfies the Hölder condition, the integral (2.118) with a singularity of order two in the two-dimensional space has a Cauchy Principal value.

3 Transformation of Differential Equations to Integral Equations

There exist several methods for transforming differential equations decribing a boundary value problem or an initial/boundary value problem to an equivalent representation by integral equations. Two are essentially different when regarding their basic ideas and result also in quite different formulations: the so-called *direct boundary integral* equation method and the *indirect boundary integral* equation method. Both derivations will be described by some representative examples.

3.1 Introductory 1-d problems: Transformation of ordinary differential equations

In R^1, not only the transformation of the considered differential equations to an integral equation representation, but also their solution can be found in many cases analytically exact. Hence, it makes sense to consider some typical examples in order to clarify some essentials, e.g., the importance of the fundamental solution of the analysed basic differential equations.

3.1.1 Integral equations by direct integration

Here, some simple integral equations are derived by performing straightforward integrations in order to show the equivalence of the formulations and to point-out the importance of the respective fundamental solution for these integral equation representations.

3.1.1.1 First order ordinary differential equation.

The most simple differential equation is certainly

$$\frac{d}{dx}y(x) = f(x) \tag{3.1}$$

where $f(x)$ is given, and the solution is unique by the initial condition $y(a) = y_0$ in an arbitrary point $x = a$ and shall be considered to be defined in a certain intervall $[a, b]$. Integrating of both sides of the equation (3.1) gives

$$y(x) = \left[\int f(\bar{x})d\bar{x}\right]_{\bar{x}=x} + c \tag{3.2}$$

When one satisfies the prescribed initial condition, i.e.,

$$y(a) = y_0 = \left[\int f(\bar{x}) d\bar{x} \right]_{\bar{x}=a} + c \tag{3.3}$$

one finds the constant c as

$$c = y_0 - \left[\int f(\bar{x}) d\bar{x} \right]_{\bar{x}=a} \tag{3.4}$$

and the 'solution' of the first order ordinary differential equation (3.1) with the initial condition $y(a) = y_0$ to be

$$y(x) = \int_a^x f(\bar{x}) d\bar{x} + y_0 \tag{3.5}$$

This example looks trivial. But, taking additionally into account that x is restricted to a one-dimensional domain whose boundary consists of the two endpoints of the closed intervall $[a, b]$, and introducing the Heaviside function $H(x - \bar{x})$ (see (2.58))

$$H(x - \xi) = \left\{ \begin{array}{c} 1 \\ 0 \end{array} \text{ für } \begin{array}{c} x > \bar{x} \\ x < \bar{x} \end{array} \right. \tag{3.6}$$

the solution (3.5) may also be written as an integral over the whole definition domain $\Omega = [a, b]$:

$$y(x) = \int_a^b H(x - \bar{x}) f(\bar{x}) d\bar{x} + y_0 \tag{3.7}$$

The kernel $H(x-\bar{x})$ of the above integral operator is obviously (see (2.59)) the fundamental solution $y^*(x, \bar{x})$ of the considered differential equation (3.1).

3.1.1.2 Second order ordinary differential equations.

Transfering the idea of the integral equation formulation (3.7) to the differential equation of second order

$$\frac{d^2}{dx^2} y(x) = f(x) \tag{3.8}$$

with the two initial conditions

$$y(a) = y_0 \text{ and } \left. \frac{d}{dx} y(x) \right|_{x=a} = y'(a) = y_0' \tag{3.9}$$

and defined on the domain $\Omega = [a, b]$, its solution should be given with its fundamental solution $y^*(x, \bar{x})$ and the initial conditions (3.9) as

$$y(x) = \int_a^b y^*(x, \bar{x}) f(\bar{x}) d\bar{x} + y_0 + y_0'(x - a) \tag{3.10}$$

Certainly, one has to know the adequate fundamental solution of the differential equation (3.8). Since the fundamental solution of the first order differential equation (3.1) is the Heaviside function $H(x-\bar{x})$, one has only to know what is the result of integrating $H(x-\bar{x})$. For this purpose, it advantageous to represent the Heaviside function as 'cut polynomial' of zero degree:

$$H(x-\bar{x}) = [x-\bar{x}]_+^0 \tag{3.11}$$

In this form, it is easy to integrate the Heaviside function and to find the fundamental solution of (3.8):

$$y^*(x,\bar{x}) = [x-\bar{x}]_+^1 = (x-\bar{x})H(x-\bar{x}) \tag{3.12}$$

This is known as fundamental solution of the bar equation (2.75). Another possible fundamental solution is (2.77)

$$\begin{aligned} y^*(x,\bar{x}) &= \frac{r}{2} = \frac{\mid x-\bar{x} \mid}{2} = \frac{1}{2}(x-\bar{x})\text{sign}(x-\bar{x}) \qquad (3.13) \\ &= \frac{1}{2}(x-\bar{x})\left[2H(x-\bar{x})-1\right] = (x-\bar{x})H(x-\bar{x}) - \frac{1}{2}(x-\bar{x}) \end{aligned}$$

because the additional linear term $-\frac{1}{2}(x-\bar{x})$ is only a trivial solution of the homogeneous differential equation (3.8).

It is not easy to recognize that the expression (3.10) is really the solution of the differential equation (3.8), but this can be shown by two straightforward integrations and a little more trickery transformation of the double integral into a single integral.

One integration of both sides of the differential equation (3.8) gives

$$\frac{dy(x)}{dx} = y_0^{'} + \int_a^x f(\bar{x})d\bar{x} \tag{3.14}$$

satisfying the initial condition $y^{'}(a) = y_0^{'}$, and a second produces

$$y(x) = y_0 + (x-a)y_0^{'} + \int_a^x \left[\int_a^s f(\bar{x})d\bar{x}\right] ds \tag{3.15}$$

the further constant of integration having been taken so that $y(a) = y_0$.

Simplification of the double integral in (3.15) follows on using the result of the more general formula

$$\int_a^x \left[\int_a^s G(\bar{x},s)d\bar{x}\right] ds = \int_a^x \left[\int_{\bar{x}}^x G(\bar{x},s)ds\right] d\bar{x} \tag{3.16}$$

for which it is sufficient that $G(\bar{x},s)$ be a continous function of both variables. To establish (3.16) note that the repeated integral on the left hand side is evaluated over a triangular region of the $\bar{x}-s$ plane, first the inner integral at a fixed s from $\bar{x}=a$ to $\bar{x}=s$ and the outer integral then runs from $s=a$ to $s=x$. On reversing the integration order, the same triangular region must be covered. This is achieved by integrating from $s=\bar{x}$ to $s=x$ at a fixed $\bar{x}$, followed by integration with respect to $\bar{x}$ from $\bar{x}=a$ to $\bar{x}=x$.

3.1.1.3 Exercise 4: Reversed order integrations

a) Check the formula (3.16) for the integrand $G(\bar{x}, s) = \bar{x}^3 s$.

b) Derive a reversed order integration form for the double integral

$$\int_a^x \left[\int_s^b G(\bar{x}, s) d\bar{x} \right] ds, \quad x \in [a, b],$$

and check it for the integrand $G(\bar{x}, s) = \bar{x}^3 s$.

Applying the formula (3.16) to the double integral in (3.15) gives

$$\int_a^x \left[\int_a^s f(\bar{x}) d\bar{x} \right] ds = \int_a^x \left[\int_{\bar{x}}^x f(\bar{x}) ds \right] d\bar{x} = \int_a^x f(\bar{x}) \left[\int_{\bar{x}}^x ds \right] d\bar{x} = \int_a^x f(\bar{x})(x - \bar{x}) d\bar{x}$$

and the integral representation (3.15) is simplified to

$$y(x) = y_0 + (x - a) y_0' + \int_a^x (x - \bar{x}) f(\bar{x}) d\bar{x} \tag{3.17}$$

or, by introducing the Heaviside function to correctly perform the integration over the whole intervall

$$y(x) = y_0 + (x - a) y_0' + \int_a^b (x - \bar{x}) H(x - \bar{x}) f(\bar{x}) d\bar{x} \tag{3.18}$$

This is obviously the same solution as given in (3.10).

3.1.1.4 Exercise 5: Integral equation by straightforward integrations

The second order differential equation (3.8) has to satisfy the two boundary conditions

$$y(a) = y_0 \quad \text{and} \quad \left. \frac{d}{dx} y(x) \right|_{x=b} = y'(b) = y_1'$$

Derive the solution of the boundary value problem by straightforward integrations and transform the resulting double integral into single integrals applying the formula derived in Exercise 2.

3.1.2 Direct integral equations by the method of weighted residuals

In general, the above described proceeding of transforming differential equations into integral equations by straightforward integrations is not applicable. Therefore, a general methodology - the method of weighted residuals - is now introduced where, for a better understanding, the same simple second order ordinary differential equation (3.8) is considered again first.

The idea of this method is as follows: One considers the residual which remains when an approximative solution is inserted in the differential equation, multiplies this residual with certain (known) weighting functions, and

demands that the integral of this product over the problem domain disappears, i.e., is zero.

Then, all differentiations acting on the unknown states of the differential equation are shifted through integration by parts to act on the known weighting functions. If the chosen weighting function is the fundamental solution of the actually considered differential equation, one obtains an equivalent integral equation formulation of the boundary value problem.

It can be used to determine the unknown boundary reactions of the problem, and, when these are found, also the sought solution of the considered differential equation at any arbitrary interior point.

3.1.2.1 Transformation of Poisson or Laplace equations

The second order ordinary differential equation (3.8) is the one-dimensional representation of the so-called *Poisson*'s or *Laplace* equation which has, as shown above (see (3.12) and (3.13)), the fundamental solution $(\xi - x)H(\xi - x)$ or equivalently $\frac{1}{2}r = \frac{1}{2} \mid x - \xi \mid$. As physically meaningful examples, the equations from Euler-Bernoulli' theory of elastic beams are considered.

3.1.2.1.1 Bending deflection of elastic beams The deflection $w(x)$ of an elastic beam under a prescribed bending moment distribution $M(x)$ has to satisfy the inhomogeneous differential equation of second order:

$$\frac{d^2}{dx^2}w(x) = -\frac{M(x)}{EI}, \ \Omega = \{ \ x \,|x \in [a, b], \ a - b = l\} \tag{3.19}$$

Following the above advices, this equation is multiplied with the fundamental solution $w^*(x, \xi) = \frac{1}{2}r$ as adequate weighting function, integrated over the problem domain, i.e., over the beam length l from $x = a$ to $x = b$, and the result is demanded to be zero:

$$\int_a^b \left(\frac{d^2 w(x)}{dx^2} + \frac{M(x)}{EI} \right) w^*(x, \xi) dx = 0 \tag{3.20}$$

or

$$\int_a^b \frac{d^2 w(x)}{dx^2} w^*(x, \xi) dx = - \int_a^b \frac{M(x)}{EI} w^*(x, \xi) dx \tag{3.21}$$

Now, the left hand side of (3.21) has to be integrated by parts twice in order to shift the two differentiation in the 'domain' integral from the unknown $w(x)$ to the known fundamental solution $w^*(x,\xi)$:

$$\begin{aligned}\int_a^b \frac{d^2w(x)}{dx^2}w^*(x,\xi)dx &= \left[\frac{dw(x)}{dx}w^*(x,\xi)\right]_a^b - \int_a^b \frac{dw(x)}{dx}\frac{\partial w^*(x,\xi)}{\partial x}dx \\ &= \left[\frac{dw(x)}{dx}w^*(x,\xi) - w(x)\frac{\partial w^*(x,\xi)}{\partial x}\right]_a^b \\ &\quad + \int_a^b w(x)\frac{\partial^2 w^*(x,\xi)}{\partial x^2}dx \end{aligned} \tag{3.22}$$

Since the second derivative of the fundamental solution (2.77) $w^*(x,\xi) = \frac{1}{2}r = \frac{1}{2} \mid x-\xi \mid$ gives the Dirac function $\delta(x,\xi)$, the result of the respective domain integral is with $\xi \in [a,b]$ simply the value of w at ξ (see (2.52)):

$$\int_a^b w(x)\frac{\partial^2 w^*(x,\xi)}{\partial x^2}dx = \int_a^b w(x)\delta(x,\xi)dx = w(\xi) \tag{3.23}$$

Hence, with (3.23) and (3.22), the equation (3.21) is transformed into the boundary integral representation

$$w(\xi) = -\left[\frac{dw(x)}{dx}w^*(x,\xi) - w(x)\frac{\partial w^*(x,\xi)}{\partial x}\right]_a^b - \int_a^b \frac{M(x)}{EI}w^*(x,\xi)dx \tag{3.24}$$

Explicitly, with (2.78), i.e.,

$$\frac{\partial w^*(x,\xi)}{\partial x} = H(x-\xi) - \frac{1}{2} = \frac{1}{2}\text{sign}(x-\xi) \tag{3.25}$$

the evaluation of (3.22) gives:

$$\begin{aligned}\int_0^l \frac{d^2w(x)}{dx^2}w^*(x,\xi)dx &= w'(b)\frac{\mid b-\xi \mid}{2} - w(b)\frac{1}{2}\text{sign}(b-\xi) \\ &\quad -w'(a)\frac{\mid a-\xi \mid}{2} + w(a)\frac{1}{2}\text{sign}(a-\xi) + w(\xi) \\ &= w'(b)\frac{(b-\xi)}{2} - w(b)\frac{1}{2} - w'(a)\frac{\xi-a}{2} - w(a)\frac{1}{2} + w(\xi) \end{aligned} \tag{3.26}$$

Hence, the equation (3.24), the integral representation of the differential equation (3.19), can also be expressed as:

$$w(\xi) = \frac{1}{2}w(a) + \frac{\xi-a}{2}w'(a) + \frac{1}{2}w(b) - \frac{(b-\xi)}{2}w'(b) - \int_0^l \frac{M(x)}{EI}\frac{1}{2} \mid x-\xi \mid dx \tag{3.27}$$

It is valid for all interior points $\xi \in [a, b]$ and all combinations of boundary conditions, but, before it is possible to evaluate this expression, all unknown boundary reactions must be determined.

For this purpose, the point ξ has to be shifted on the boundary, i.e. here, at the two boundary points $\xi = a$ and $\xi = b$, to obtain two equations for the two unknown boundary values. This gives the following equation system ($b - a = l$):

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & -1 & l \\ -1 & -l & 1 & 0 \end{bmatrix} \begin{bmatrix} w(a) \\ w^{'}(a) \\ w(b) \\ w^{'}(b) \end{bmatrix} = -\frac{1}{2EI}\int_a^b \begin{bmatrix} (x-a)M(x) \\ (b-x)M(x) \end{bmatrix} dx \tag{3.28}$$

Dependent on the actually prescribed boundary conditions, the corresponding colums have to be multiplied with the respective known values and transfered to the right hand side. The solution of the resulting system delivers the unknown boundary reactions which are necessary for the evaluation of the integral representation (3.27).

In order to compare with the already solved problem, the initial value problem with the conditions (3.9)

$$w(a) = w_0 \text{ and } w^{'}(a) = w_0^{'} \tag{3.29}$$

and $f(x) = -M(x)/EI$ is considered here again. Then, the actual algebraic equation system is

$$\begin{bmatrix} -1 & l \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w(b) \\ w^{'}(b) \end{bmatrix} = \int_0^l \begin{bmatrix} (x-a)f(x) \\ (b-x)f(x) \end{bmatrix} dx - \begin{bmatrix} 1 & 0 \\ -1 & -l \end{bmatrix} \begin{bmatrix} w_0 \\ w_0^{'} \end{bmatrix} \tag{3.30}$$

from which one obtains

$$w(b) = \int_a^b (b-x)f(x)dx + w_0 + lw_0^{'} \tag{3.31}$$

$$\begin{aligned} w^{'}(b) &= \frac{1}{l}\left(\int_a^b (x-a)f(x)dx - w_0 + w(b)\right) \\ &= \int_a^b f(x)dx + w_0^{'} \end{aligned} \tag{3.32}$$

Inserting these boundary reactions in the integral representation (3.27) of the solution

$$\begin{aligned}
w(\xi) &= \frac{1}{2}w_0 + \frac{\xi - a}{2}w_0^{'} + \frac{1}{2}w(b) - \frac{(b-\xi)}{2}w^{'}(b) + \int\limits_0^l f(x)\frac{\mid x-\xi \mid}{2}dx \\
&= \frac{1}{2}w_0 + \frac{\xi - a}{2}w_0^{'} + \frac{1}{2}\left(\int\limits_0^l (b-x)f(x)dx + w_0 + lw_0^{'}\right) \\
&\quad -\frac{(b-\xi)}{2}\left(\int\limits_0^l f(x)dx + w_0^{'}\right) + \int\limits_0^l f(x)\frac{\mid x-\xi \mid}{2}dx \\
&= w_0 + (\xi - a)w_0^{'} + \int\limits_0^l f(x)\frac{1}{2}\left(\xi - x + \mid x - \xi \mid\right)dx \\
&= w_0 + (\xi - a)w_0^{'} + \int\limits_0^l f(x)(\xi - x)H(\xi - x)dx
\end{aligned} \tag{3.33}$$

where the equality of $\frac{1}{2}\left(\xi - x + \mid x - \xi \mid\right)$and $(\xi - x)H(\xi - x)$ has been taken into account. The final line in (3.33) is exactly the same expression for the solution of the initial value problem as given in (3.18) when one recognizes that ξ and x are there x and $\bar{x}$, respectively.

Remark: An essential difference of the direct boundary integral formulation obtain via the method of weighted residuals in comparison to the integral solution by straightforward integrations is that one has to determine first the unknown boundary reactions before one can evaluate the expression for the solution at arbitrary interior points.

3.1.2.1.2 Exercise 6: Beam deflection under prescribed moments Use the above system (3.28) to solve the boundary value problem with the prescribed conditions $w(a) = w_0$ and $w^{'}(b) = w_1^{'}$ which corresponds with $y \triangleq w$ and $f(x) = -M(x)/EI$ to the problem of Exercise 3.

3.1.2.1.3 Exercise 7: Bending moment of an elastic beams under transversal loading Transfer the above solution (3.27) and the system (3.28) for determining the boundary reactions to the differential equation for the bending moment $M(x)$ of an elastic beam under the prescribed transversal loading $q(x)$:

$$\frac{d^2M(x)}{dx^2} = -q(x) \tag{3.34}$$

Note that the shear force $Q(x)$is the first derivative of the bending moment, i.e., $Q(x) = M^{'}(x) = dM(x)/dx$.

3.1.2.1.4 Exercise 8: Axial displacement of an elastic bar Transfer the above solution (3.27) and the system (3.28) for determining the boundary reactions to the differential equation for the axial displacement $u(x)$ of an elastic bar of length l with sectional area A and Young's modulus E under the prescribed axial loading $p(x)$:

$$\frac{d^2u(x)}{dx^2} = -\frac{p(x)}{EA} \tag{3.35}$$

Note that the resultant axial force $N(x)$ is related to the axial displacement $u(x)$ via $N(x) = EAu^{'}(x)$.

3.1.2.1.5 Exercise 9: Bar stretching under axial loadings Solve the stretching problem of an elastic bar, which is fixed at $x = a = 0$, i.e, $u(0) = 0$ and has a free ending at $x = b = l$, i.e., $N(l) = 0$, with the integral equation system and the solution expression determined in Exercise 8. The prescribed axial loading is $p(x) = p_0 \frac{x}{l}$.

3.1.2.2 Transformation of Helmholtz equations

The till now considered differential equations contained only first or second order derivatives of the sought solution. When the sought solution function multiplied by a constant factor is added to or substracted from the second derivative term, e.g.,

$$\frac{d^2w(x)}{dx^2} + k^2w(x) = f(x) \quad \text{or} \quad \frac{d^2w(x)}{dx^2} - h^2w(x) = f(x) \tag{3.36}$$

this differential equation is called of Helmholtz type. When one prefers symmetric forms, the respective fundamental solutions are (see, (2.89) and (2.93), respectively, and corresponding Remark 1 there)

$$w^*(x,\xi) = w^*(r) = \frac{1}{2k}\sin(kr) \text{ with } k \neq 0, \text{ real, } r = |x-\xi| \tag{3.37}$$

$$w^*(x,\xi) = w^*(r) = \frac{1}{2h}\sinh(hr) \text{ with } h \neq 0, \text{ real, } r = |x-\xi| \tag{3.38}$$

or, instead of the for $r \to \infty$ divergent form (3.38), the convergent one (2.97)

$$u^*(x,\xi) = u^*(r) = \frac{-1}{2h}e^{-hr} \tag{3.39}$$

3.1.2.2.1 Stationary longitudinal waves in an elastic bar The dynamic equilibrium of a bar element with the cross section A and material density ρ under an axial loading $p(x,t)$ and a londitudinal accelleration $\ddot{u}(x) = \partial^2 u(x,t)/\partial t^2$ is described by

$$\frac{\partial N(x,t)}{\partial x} = -p(x,t) + \rho A \frac{\partial^2 u(x,t)}{\partial t^2} \tag{3.40}$$

where the axial resultant force $N(x,t)$ is related to the longitudinal displacement $u(x,t)$ by

$$N(x,t) = EA\frac{\partial u(x,t)}{\partial x} = EAu'(x,t) \tag{3.41}$$

Connecting both equations and assuming constant cross section A and modulus of elasticity E yields the basic differential equation

$$EA\frac{\partial^2 u(x,t)}{\partial x^2} = -p(x,t) + \rho A\frac{\partial^2 u(x,t)}{\partial t^2} \tag{3.42}$$

or, introducing the longitudinal wave speed $c_L = \sqrt{E/\rho}$,

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c_L^2}\frac{\partial^2 u(x,t)}{\partial t^2} = -\frac{p(x,t)}{EA} \tag{3.43}$$

For time-harmonic loadings with the excitation frequency ω

$$p(x,t) = p(x)e^{\mathrm{i}\omega t} \tag{3.44}$$

the response can also be assumed to be time-harmonic:

$$u(x,t) = u(x)e^{\mathrm{i}\omega t} \tag{3.45}$$

The result is an equation which is not longer time-dependent and is of Helmholtz type

$$\frac{d^2 u(x)}{dx^2} + \kappa^2 u(x) = -\frac{p(x)}{EA} \tag{3.46}$$

The ratio $\kappa = \omega/c_L$ is the so-called *wave number*.

As we already know, the transformation of this differential equation (3.46) to an equivalent integral equation may be performed by integrations by parts of the integral of the weighted residual over the problem domain, i.e. here, over the bar length l:

$$\int_0^l \left[\frac{d^2 u(x)}{dx^2} + \kappa^2 u(x) + \frac{p(x)}{EA}\right] u^*(x,\xi)dx = 0 \tag{3.47}$$

where the fundamental solution $u^*(x,\xi)$ of the differential equation is taken as special weigthing function.

The two integrations by parts of the first term in (3.47) gives

$$\begin{aligned}\int_0^l \frac{d^2 u(x)}{dx^2}u^*(x,\xi)dx &= [u'(x)u^*(x,\xi)]_0^l - \int_0^l u'(x)\frac{\partial u^*(x,\xi)}{\partial x}dx \\ &= \left[u'(x)u^*(x,\xi) - u(x)\frac{\partial u^*(x,\xi)}{\partial x}\right]_0^l + \int_0^l u(x)\frac{\partial^2 u^*(x,\xi)}{\partial x^2}dx\end{aligned} \tag{3.48}$$

such that the complete transformation of (3.47) becomes

$$\left[u'(x)u^*(x,\xi) - u(x)\frac{\partial u^*(x,\xi)}{\partial x}\right]_0^l + \int_0^l \left[\frac{\partial^2 u^*(x,\xi)}{\partial x^2} + \kappa^2 u^*(x,\xi)\right] u(x)dx = -\int_0^l \frac{p(x)}{EA} u^*(x,\xi)dx \tag{3.49}$$

Here, the adequate fundamental solution is (see (3.38)

$$u^*(x,\xi) = u^*(r) = \frac{1}{2k}\sin(kr) \tag{3.50}$$

Its first and second derivative, respectively, is ($2H(x-\xi) - 1 = \text{sign}(x-\xi)$)

$$\frac{\partial u^*(x,\xi)}{\partial x} = \frac{1}{2}\cos(kr)\frac{\partial r}{\partial x} = \frac{1}{2}\cos(kr)[2H(x-\xi) - 1] \tag{3.51}$$

$$\frac{\partial^2 u^*(x,\xi)}{\partial x^2} = -\frac{k}{2}\sin(kr) + \cos(kr)\delta(x-\xi) \tag{3.52}$$

such that

$$\int_0^l \left[\frac{\partial^2 u^*(x,\xi)}{\partial x^2} + \kappa^2 u^*(x,\xi)\right] u(x)dx = \int_0^l \cos(kr)\delta(x-\xi)u(x)dx = u(\xi) \quad \text{for } \xi \in [0,l]$$

Thus, the final result of (3.49) is the following integral expression for the axial displacement at an arbitrary point $\xi \in [0,l]$:

$$\begin{aligned}
u(\xi) &= -\left[u'(x)u^*(x,\xi) - u(x)\frac{\partial u^*(x,\xi)}{\partial x}\right]_0^l - \int_0^l \frac{p(x)}{EA}u^*(x,\xi)dx \\
&= -\frac{1}{2k}\text{sin}k(l-\xi)u'(l) + \frac{1}{2}\text{cos}k(l-\xi)\text{sign}(l-\xi)u(l) \\
&\quad + \frac{1}{2k}\text{sin}(k\xi)u'(0) - \frac{1}{2}\text{cos}(k\xi)\text{sign}(-\xi)u(0) - \int_0^l \frac{p(x)}{EA}u^*(x,\xi)dx \\
&= -\frac{1}{2EAk}\text{sin}k(l-\xi)N(l) + \frac{1}{2}\text{cos}k(l-\xi)u(l) \\
&\quad + \frac{1}{2EAk}\text{sin}(k\xi)N(0) + \frac{1}{2}\text{cos}(k\xi)u(0) - \int_0^l \frac{p(x)}{EA}\frac{1}{2k}\sin(kr)dx
\end{aligned} \tag{3.53}$$

where the relation $N(x) = EAu'(x)$ was applied in order to introduce the adequate boundary state $N(x)$.

Since two of the four boundary values are unknown, one needs two equation to determine their values. They are obtain by *collocation*, i.e., evaluation of the equation (3.53) at the two boundary points (both equations have been multiplied by 2):

$$\begin{aligned}\xi &= 0: \; u(0) - \cos(kl)\, u(l) + \frac{1}{EAk}\sin(kl)\, N(l) = -\int_0^l \frac{p(x)}{EAk}\sin(kx)dx \\ \xi &= l: \; -\cos(kl)\, u(0) + u(l) - \frac{1}{EAk}\sin(kl)\, N(0) = -\int_0^l \frac{p(x)}{EAk}\sin k(l-x)dx \end{aligned} \tag{3.54}$$

or in matrix notation

$$\begin{bmatrix} 1 & 0 & -\cos(kl) & \sin(kl) \\ -\cos(kl) & -\sin(kl) & 1 & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ \frac{N(0)}{EAk} \\ u(l) \\ \frac{N(l)}{EAk} \end{bmatrix} = -\frac{1}{EAk}\int_0^l \begin{bmatrix} p(x)\sin(kx) \\ p(x)\sin k(l-x) \end{bmatrix} dx \tag{3.55}$$

3.1.2.2.2 Exercise 10: Torsional twist of an elastic bar Transfer the above integral form (3.53) of the solution and the equation system (3.54) for determining the boundary reactions to the differential equation for the angular twist change $\vartheta(x) = d\theta/dx$ of an elastic bar of length l under a torsional moment $M_T(x)$:

$$\frac{d^2\vartheta(x)}{dx^2} - h^2\vartheta(x) = -\frac{M_T(x)}{EC_T} \tag{3.56}$$

The constant factor $h^2 = \frac{GI_T}{EC_T}$ is the ratio of the torsional stiffness GI_T and the warping resistance EC_T.

3.1.2.3 Transformation of a Bilaplacian equation

When the 1-d form of the Laplace operator $\Delta = \mathrm{d}^2/\mathrm{d}x^2$ is applied twice to a sought function $w(x)$, the so-called Bilaplacian or Biharmonic equation is obtained:

$$\mathcal{L}(w) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\frac{\mathrm{d}^2}{\mathrm{d}x^2}w(x)) = f(x) \tag{3.57}$$

Its fundamental solution is found by integrating two times the fundamental solution $w^*(x,\xi) = \frac{1}{2}r$ of the Poisson equation (2.76) which gives

$$w^*(x,\xi) = \frac{1}{12}r^3 \tag{3.58}$$

This can easily be checked by straight-forward differentiations

$$\frac{\partial w^*(x,\xi)}{\partial x} = \frac{1}{4}r^2\frac{\partial r}{\partial x} = \frac{1}{4}r^2\left(2H(x-\xi)-1\right) \tag{3.59}$$

$$\frac{\partial^2 w^*(x,\xi)}{\partial x^2} = \frac{1}{2}r\left(\frac{\partial r}{\partial x}\right)^2 + \frac{1}{2}r^2\delta(x-\xi) = \frac{1}{2}r \tag{3.60}$$

$$\frac{\partial^3 w^*(x,\xi)}{\partial x^3} = \frac{1}{2}\frac{\partial r}{\partial x} = H(x-\xi) - \frac{1}{2} \tag{3.61}$$

$$\frac{\partial^4 w^*(x,\xi)}{\partial x^4} = \delta(x-\xi) \tag{3.62}$$

Remark: The term $\frac{1}{2}r^2\delta(x-\xi)$ in the second derivative can be neglected since it is zero due to $r=0$ for $x=\xi$.

Now, this most simple form of a 4th order differential equation will be tranformed into an integral formulation for its solution $w(x)$ and for other related states, e.g., $w^{'}(x)$.

3.1.2.3.1 The Euler-Bernoulli beam In the Euler-Bernoulli theory for the bending of elastic beams, the deflection $w(x)$ is described by the 4th order differential equation

$$EI\frac{d^4w(x)}{dx^4} = q(x) \tag{3.63}$$

where EI means its bending stiffness. For a unique solution, four boundary conditions have to be prescribed where at each boundary point two boundary values are known corresponding to the actual support while the other two are unknown reactions, e.g.,

$$\begin{array}{rcc} \text{for clamped endings} & w=0 & w^{'}=0 \\ \text{for a free ending} & M=0 & Q=0 \\ \text{for a simple support} & w=0 & M=0 \end{array} \tag{3.64}$$

Integral equation for the beam deflection The method of weighted residual postulates

$$\int_0^l \left(EI\frac{d^4w(x)}{dx^4} - q(x)\right) w^*(x,\xi)dx = 0 \tag{3.65}$$

or

$$\int_0^l EI\frac{d^4w(x)}{dx^4}\, w^*(x,\xi)dx = \int_0^l q(x)\, w^*(x,\xi)dx \tag{3.66}$$

where the weighting function $w^*(x,\xi)$ has to be the fundamental solution of the differential equation (3.63). This is obviously obtained from (3.58) by simply dividing by EI

$$w^*(x,\xi) = \frac{r^3}{12EI} \tag{3.67}$$

Its derivatives differ from (3.59) to (3.62) only by the factor $1/EI$, e.g.,

$$\frac{\partial^3 w^*(x,\xi)}{\partial x^3} = \frac{1}{2EI}\frac{\partial r}{\partial x} = \frac{1}{2EI}[2H(x-\xi)-1] \tag{3.68}$$

The procedure for deriving the boundary integral form is analogous to that in the case of the differential equations of second order: one has only to integrate by parts four times instead of two times. Having in mind that

$$EI\frac{d^2w(x)}{dx^2} = -M(x), \quad EI\frac{d^3w(x)}{dx^3} = -Q(x) \tag{3.69}$$

one obtains by the first integration by parts

$$\int_0^l EI\frac{d^4w(x)}{dx^4}w^*(x,\xi)dx = \left[EI\frac{d^3w(x)}{dx^3}w^*(x,\xi)\right]_0^l - \int_0^l EI\frac{d^3w(x)}{dx^3}\frac{\partial w^*(x,\xi)}{\partial x}dx$$

by two integrations by parts

$$\begin{aligned}\int_0^l EI\frac{d^4w(x)}{dx^4}w^*(x,\xi)dx &= \left[-Q(x)w^*(x,\xi) - EI\frac{d^2w(x)}{dx^2}\frac{\partial w^*(x,\xi)}{\partial x}\right]_0^l \\ &\quad + \int_0^l EI\frac{d^2w(x)}{dx^2}\frac{\partial^2 w^*(x,\xi)}{\partial x^2}dx\end{aligned}$$

by three integrations by parts

$$\begin{aligned}\int_0^l EI\frac{d^4w(x)}{dx^4}w^*(x,\xi)dx &= \left[-Q(x)w^*(x,\xi) + M(x)w^{*'}(x,\xi) - w'(x)M^*(x,\xi)\right]_0^l \\ &\quad - \int_0^l EI\frac{dw(x)}{dx}\frac{\partial^3 w^*(x,\xi)}{\partial x^3}dx\end{aligned}$$

and by the last forth integration by parts

$$\begin{aligned}\int_0^l EI\frac{d^4w(x)}{dx^4}w^*(x,\xi)dx &= \left[-Q(x)w^*(x,\xi) + M(x)w^{*'}(x,\xi) - w'(x)M^*(x,\xi)\right]_0^l \\ &\quad - \left[EIw(x)\frac{\partial^3 w^*(x,\xi)}{\partial x^3}\right]_0^l + \int_0^l EIw(x)\frac{\partial^4 w^*(x,\xi)}{\partial x^4}dx \\ &= \left[\begin{array}{c}-Q(x)w^*(x,\xi) + M(x)w^{*'}(x,\xi) \\ -w'(x)M^*(x,\xi) + w(x)Q^*(x,\xi)\end{array}\right]_0^l \\ &\quad + \int_0^l EIw(x)\frac{\delta(x-\xi)}{EI}dx\end{aligned}$$

Hence, taking the 'filtering' effect of the Dirac function into account, one gets

$$\begin{aligned} w(\xi) \;=\; & \Big[Q(x)w^*(x,\xi) - M(x)w^{*'}(x,\xi) + w^{'}(x)M^*(x,\xi) - w(x)Q^*(x,\xi)\Big]_0^l \\ & + \int_0^l q(x)\, w^*(x,\xi)dx \end{aligned} \tag{3.70}$$

or explicitly

$$\begin{aligned} w(\xi) \;=\; & Q(l)w^*(l,\xi) - Q(0)w^*(0,\xi) - w(l)Q^*(l,\xi) + w(0)Q^*(0,\xi) \\ & - M(l)w^{*'}(l,\xi) + w^{'}(l)M^*(l,\xi) + M(0)w^{*'}(0,\xi) - w^{'}(0)M^*(0,\xi) \\ & + \int_0^l q(x)\, w^*(x,\xi)dx \end{aligned} \tag{3.71}$$

In this equation, four of the eight boundary values are known and the other four are unknown reactions. Evaluating this equation at the two boundary points, i.e., at $\xi = 0+\varepsilon$ and $\xi = l - \varepsilon$ gives the two boundary 'integral' equations

$$\frac{1}{2}w(0) - \frac{1}{2}w(l) - Q(l)\frac{l^3}{12EI} + M(l)\frac{l^2}{4EI} + w^{'}(l)\frac{l}{2} = \int_0^l q(x)\,\frac{x^3}{12EI}dx \tag{3.72}$$

$$-\frac{1}{2}w(0) + \frac{1}{2}w(l) + Q(0)\frac{l^3}{12EI} + M(0)\frac{l^2}{4EI} - w^{'}(0)\frac{l}{2} = -\int_0^l q(x)\,\frac{(x-l)^3}{12EI}dx \tag{3.73}$$

where the evaluation of the fundamental solution has given ($\varepsilon \to 0$)

$$w^*(l,0) = \frac{l^3}{12EI},\; w^*(0,0) = 0,\; w^*(l,l) = 0,\; w^*(0,l) = \frac{l^3}{12EI} \tag{3.74}$$

and for the related fundamental states

$$w^{*'}(l,0) = \frac{l^2}{4EI},\; w^{*'}(0,0) = 0,\; w^{*'}(l,l) = 0,\; w^{*'}(0,l) = -\frac{l^2}{4EI} \tag{3.75}$$

$$M^*(l,0) = -EIw^{*''}(l,0) = -\frac{l}{2},\; M^*(0,0) = 0,\; M^*(l,l) = 0,\; M^*(0,l) = -\frac{l}{2} \tag{3.76}$$

$$Q^*(l,0) = -\frac{1}{2},\; Q^*(0,0) = \frac{1}{2},\; Q^*(l,l) = -\frac{1}{2},\; Q^*(0,l) = \frac{1}{2} \tag{3.77}$$

Now, we obtained two boundary integral equations but one needs two equations more.

Integral equation for the beam slope Obviously, one gets a new integral equation for the beam slope $w^{'}(\xi)$, which means a rotation about the y-axis, when one differentiates the integral equation (3.70) with respect to the variable ξ. Since the equation (3.70) was obtained by integration by parts of the weighted residual (3.66) of the differential equation (3.63), it is clear that one can perform the differentiation with respect to ξ also directly to the weighted residual:

$$\int_0^l EI\frac{d^4w(x)}{dx^4}\,\frac{\partial w^*(x,\xi)}{\partial \xi}dx = \int_0^l q(x)\,\frac{\partial w^*(x,\xi)}{\partial \xi}dx \tag{3.78}$$

Hence, the relation (3.78) is also a weighted residual form of the beam equation (3.63) where only another weighting function is applied, namely

$$w_2^*(x,\xi) = \frac{\partial w^*(x,\xi)}{\partial \xi} = -\frac{\partial w^*(x,\xi)}{\partial x} = -\frac{1}{4EI}(x-\xi)^2\mathrm{sgn}(x-\xi) \tag{3.79}$$

with

$$\begin{aligned}
\frac{\partial w_2^*(x,\xi)}{\partial x} &= -\frac{\partial^2 w^*(x,\xi)}{\partial x^2} = -\frac{r}{2EI} \\
\frac{\partial^2}{\partial x^2}w_2^*(x,\xi) &= \frac{-1}{2EI}\frac{\partial r}{\partial x} = \frac{-1}{2EI}[2H(x-\xi)-1] \\
\frac{\partial^3}{\partial x^3}w_2^*(x,\xi) &= \frac{-1}{EI}\delta(x-\xi)
\end{aligned}$$

Then, integration by parts yields at first

$$\begin{aligned}
\int_0^l EI\frac{d^4w(x)}{dx^4}w_2^*(x,\xi)dx &= \left[EI\frac{d^3w(x)}{dx^3}w_2^*(x,\xi)\right]_0^l - \int_0^l EI\frac{d^3w(x)}{dx^3}\frac{\partial w_2^*(x,\xi)}{\partial x}dx \\
&= [-Q(x)w_2^*(x,\xi)]_0^l - \int_0^l EI\frac{d^3w(x)}{dx^3}\frac{\partial w_2^*(x,\xi)}{\partial x}dx
\end{aligned} \tag{3.80}$$

by the second integration

$$\begin{aligned}
\int_0^l EI\frac{d^4w(x)}{dx^4}w_2^*(x,\xi)dx &= \left[-Q(x)w_2^*(x,\xi) - EI\frac{d^2w(x)}{dx^2}\frac{\partial w_2^*(x,\xi)}{\partial x}\right]_0^l \\
&\quad + \int_0^l EI\frac{d^2w(x)}{dx^2}\frac{\partial^2 w_2^*(x,\xi)}{\partial x^2}dx \\
&= \left[-Q(x)w_2^*(x,\xi) + M(x)\frac{\partial w_2^*(x,\xi)}{\partial x}\right]_0^l + \int_0^l EI\frac{d^2w(x)}{dx^2}\frac{\partial^2 w_2^*(x,\xi)}{\partial x^2}dx
\end{aligned}$$

and finally already by the third integration by parts ($M_2^*(x,\xi) = -EI\partial^2 w_2^*(x,\xi)/\partial x^2$)

$$\begin{aligned}
\int_0^l EI\frac{d^4w(x)}{dx^4}w_2^*(x,\xi)dx &= \left[-Q(x)w_2^*(x,\xi) + M(x)\frac{\partial w_2^*(x,\xi)}{\partial x} + EI\frac{dw(x)}{dx}\frac{\partial^2 w_2^*(x,\xi)}{\partial x^2}\right]_0^l \\
&\quad - \int_0^l EI\frac{dw(x)}{dx}\frac{\partial^3 w_2^*(x,\xi)}{\partial x^3}dx \\
&= \left[-Q(x)w_2^*(x,\xi) + M(x)\frac{\partial w_2^*(x,\xi)}{\partial x} - \frac{dw(x)}{dx}M_2^*(x,\xi)\right]_0^l \\
&\quad + \int_0^l \frac{dw(x)}{dx}\delta(x,\xi)dx \\
&= \left[-Q(x)w_2^*(x,\xi) + M(x)\frac{\partial w_2^*(x,\xi)}{\partial x} - \frac{dw(x)}{dx}M_2^*(x,\xi)\right]_0^l \\
&\quad + \left.\frac{dw(x)}{dx}\right|_{x=\xi}
\end{aligned}$$

Consequentyl, the integral equation for $w^{'}(\xi)$ at interior points $\xi \in (0,l)$ reads as

$$w^{'}(\xi) = \left[Q(x)w_2^*(x,\xi) - M(x)\frac{\partial w_2^*(x,\xi)}{\partial x} + \frac{dw(x)}{dx}M_2^*(x,\xi)\right]_0^l + \int_0^l q(x)\, w_2^*(x,\xi)dx \quad (3.81)$$

The evaluation of this equation at the two boundary points gives the two extra equations for the determination of the four unknown boundary reactions:

for $\xi = 0 + \varepsilon$ ($\varepsilon \to 0$):

$$w^{'}(0) - \left[Q(x)w_2^*(x,\varepsilon) - M(x)\frac{\partial w_2^*(x,\varepsilon)}{\partial x} + \frac{dw(x)}{dx}M_2^*(x,\varepsilon)\right]_0^l = \int_0^l q(x)\, w_2^*(x,0)dx \quad (3.82)$$

for $\xi = l - \varepsilon$ ($\varepsilon \to 0$):

$$w^{'}(l) - \left[Q(x)w_2^*(x,l-\varepsilon) - M(x)\frac{\partial w_2^*(x,l-\varepsilon)}{\partial x} + \frac{dw(x)}{dx}M_2^*(x,l-\varepsilon)\right]_0^l = \int_0^l q(x)\, w_2^*(x,l)dx \quad (3.83)$$

When the respective function values (with $\varepsilon \to 0$) of this weighting function and of its derivatives, respectively,

$$w_2^*(l,\varepsilon) = -\frac{l^2}{4EI},\; w_2^*(0,\varepsilon) = 0,\; w_2^*(l,l-\varepsilon) = 0,\; w_2^*(0,l-\varepsilon) = \frac{l^2}{4EI} \quad (3.84)$$

$$w_2^{*'}(l,\varepsilon) = -\frac{l}{2EI},\ w_2^{*'}(0,\varepsilon) = 0,\ w_2^{*'}(l,l-\varepsilon) = 0,\ w_2^{*'}(0,l-\varepsilon) = -\frac{l}{2EI} \tag{3.85}$$

$$M_2^{*}(l,\varepsilon) = \frac{1}{2},\ M_2^{*}(0,\varepsilon) = -\frac{1}{2},\ M_2^{*}(l,l-\varepsilon) = \frac{1}{2},\ M_2^{*}(0,l-\varepsilon) = -\frac{1}{2} \tag{3.86}$$

are introduced, one obtains the two equations:

for $\xi = 0$:

$$\frac{1}{2}[w^{'}(0) - w^{'}(l)] + Q(l)\frac{l^2}{4EI} - M(l)\frac{l}{2EI} = -\int_0^l q(x)\,\frac{x^2}{4EI}dx \tag{3.87}$$

for $\xi = l$:

$$\frac{1}{2}[w^{'}(l) - w^{'}(0)] + Q(0)\frac{l^2}{4EI} + M(0)\frac{l}{2EI} = \int_0^l q(x)\,\frac{(x-l)^2}{4EI}dx \tag{3.88}$$

3.1.2.3.2 The complete system of integral equations for the deflection and the slope Althogether, the equations (3.72), (3.73), (3.87), and (3.88) result the following system (in matrix-vector notation)

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{-1}{2} & \frac{l}{2} & \frac{l^2}{4EI} & \frac{-l^3}{12EI} \\ \frac{-1}{2} & \frac{-l}{2} & \frac{l^2}{4EI} & \frac{l^3}{12EI} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{-1}{2} & \frac{-l}{2EI} & \frac{l^2}{4EI} \\ 0 & \frac{-1}{2} & \frac{l}{2EI} & \frac{l^2}{4EI} & 0 & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} w(0) \\ w^{'}(0) \\ M(0) \\ Q(0) \\ w(l) \\ w^{'}(l) \\ M(l) \\ Q(l) \end{bmatrix} = \int_0^l \begin{bmatrix} q(x)\,\frac{x^3}{12EI} \\ -q(x)\,\frac{(x-l)^3}{12EI} \\ -q(x)\,\frac{x^2}{4EI} \\ q(x)\,\frac{(x-l)^2}{4EI} \end{bmatrix} dx \tag{3.89}$$

A rearrangement of these equations, i.e., an interchanging of the first and forth line, gives a more systematic order of the coefficient matrix::

$$\begin{bmatrix} 0 & \frac{-1}{2} & \frac{l}{2EI} & \frac{l^2}{4EI} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{-1}{2} & \frac{-l}{2} & \frac{l^2}{4EI} & \frac{l^3}{12EI} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{-1}{2} & \frac{-l}{2EI} & \frac{l^2}{4EI} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{-1}{2} & \frac{l}{2} & \frac{l^2}{4EI} & \frac{-l^3}{12EI} \end{bmatrix} \begin{bmatrix} w(0) \\ w^{'}(0) \\ M(0) \\ Q(0) \\ w(l) \\ w^{'}(l) \\ M(l) \\ Q(l) \end{bmatrix} = \int_0^l \begin{bmatrix} q(x)\,\frac{(x-l)^2}{4EI} \\ -q(x)\,\frac{(x-l)^3}{12EI} \\ -q(x)\,\frac{x^2}{4EI} \\ q(x)\,\frac{x^3}{12EI} \end{bmatrix} dx \tag{3.90}$$

3.1.3 Integral formulation with Green's functions

As described in section 2.4.3, Green's functions are special fundamental solutions which additionally satisfy certain homogeneous boundary conditions, or more exactly, those boundary states of the Green's function have to be zero which are prescribed in the actual problem.

3.1.3.1 Stretching of bars

As derived in section 3.1.1.2 for general Laplace equations and explicitly given for bars in the solution of Exercise 7, the direct form of the integral equation for the axial displacement of an elastic bar of length $l = b - a$ is

$$u(\xi) = -\left[\frac{N(x)}{EA}u^*(x,\xi) - u(x)\frac{N^*(x,\xi)}{EA}\right]_a^b - \int_a^b \frac{\bar{p}(x)}{EA}u^*(x,\xi)dx \qquad (3.91)$$

There, the fundamental solution is (see, (2.77))

$$u^*(x,\xi) = \frac{1}{2}r = \frac{1}{2} \mid x - \xi \mid \qquad (3.92)$$

and satisfies the 1-dimensional Laplace equation

$$\frac{d^2u^*(x,\xi)}{dx^2} = \delta(x,\xi) \qquad (3.93)$$

When an actual boundary value problem is, e.g.,

$$\frac{d^2u(x)}{dx^2} = -\frac{\bar{p}(x)}{EA} \qquad (3.94)$$

with the boundary conditions

$$u(x=a) = \bar{u}_0 \quad \text{and} \quad N(x=b) = EA\,\frac{du(x)}{dx}\bigg|_{x=b} = \bar{N}_l \qquad (3.95)$$

the adequate Green's function $G^{uN}(x,\xi)$ of this problem has to satisfy the equation (3.93), i.e., has to be a fundamental solution, but has additionally to fulfill the homogeneous conditions

$$G^{uN}(x=a,\xi) = 0 \quad \text{and} \quad EA\,\frac{\partial G^{uN}(x,\xi)}{\partial x}\bigg|_{x=b} = 0 \qquad (3.96)$$

Hence, as easily can be found, this special Green's function is given as

$$G^{uN}(x,\xi) = u^*(x,\xi) - u^*(a,\xi) - \frac{\partial u^*(x,\xi)}{\partial x}\bigg|_{x=b}(x-a) \qquad (3.97)$$

Now, applying in the integral equation (3.91) instead of $u^*(x,\xi)$ the adequate Green's function (3.97) and taking the actual boundary conditions (3.95) and the corresponding homogeneous boundary conditions of the Green's function (3.96) into account, the solution of the boundary value problem, the axial displacement at any interior point ξ $(a \leq \xi \leq b)$ is directly found to be

$$u(\xi) = -\frac{\bar{N}_l}{EA} G^{uN}(b,\xi) - \bar{u}_0 \left. \frac{\partial G^{uN}(x,\xi)}{\partial x} \right|_{x=a} - \int_a^b \frac{\bar{p}(x)}{EA} G^{uN}(x,\xi) dx \tag{3.98}$$

Since with a$\leq \xi \leq b$

$$\begin{aligned} G^{uN}(b,\xi) &= u^*(b,\xi) - u^*(a,\xi) - \left. \frac{\partial u^*(x,\xi)}{\partial x} \right|_{x=b} (b-a) \\ &= \frac{1}{2} \mid b-\xi \mid - \frac{1}{2} \mid a-\xi \mid - \frac{1}{2} \mathrm{sign}(b-\xi) l \\ &= \frac{1}{2} (b-\xi-\xi+a-l) \\ &= a-\xi \end{aligned} \tag{3.99}$$

and

$$\begin{aligned} \left. \frac{\partial G^{uN}(x,\xi)}{\partial x} \right|_{x=a} &= \left. \frac{\partial u^*(x,\xi)}{\partial x} \right|_{x=a} - \left. \frac{\partial u^*(x,\xi)}{\partial x} \right|_{x=b} \\ &= \frac{1}{2} \mathrm{sign}(a-\xi) - \frac{1}{2} \mathrm{sign}(b-\xi) \\ &= -1 \end{aligned} \tag{3.100}$$

the solution (3.98) is explicitly

$$\begin{aligned} u(\xi) &= -\frac{\bar{N}_l}{EA}(a-\xi) + \bar{u}_0 - \int_a^b \frac{\bar{p}(x)}{EA} \frac{1}{2} \left(\mid x-\xi \mid - \mid a-\xi \mid - \mathrm{sign}(b-\xi)(x-a) \right) dx \\ &= \frac{\bar{N}_l}{EA}(\xi-a) + \bar{u}_0 - \int_a^b \frac{\bar{p}(x)}{EA} \frac{1}{2} \left(\mid x-\xi \mid -(\xi-a) - (x-a) \right) dx \\ &= \frac{\bar{N}_l}{EA}(\xi-a) + \bar{u}_0 - \int_a^\xi \frac{\bar{p}(x)}{EA} \frac{1}{2} \left(\xi-x-\xi+2a-x \right) dx \\ &\quad - \int_\xi^b \frac{\bar{p}(x)}{EA} \frac{1}{2} \left(x-\xi-\xi+2a-x \right) dx \\ &= \frac{\bar{N}_l}{EA}(\xi-a) + \bar{u}_0 - \int_a^\xi \frac{\bar{p}(x)}{EA} \left(-x+a \right) dx - \int_\xi^b \frac{\bar{p}(x)}{EA} \left(-\xi+a \right) dx \end{aligned} \tag{3.101}$$

e.g., for $\bar{p}(x) = p_0$ =const. and with $a = 0$ and $b = l$

$$\begin{aligned}
u(\xi) &= \frac{\bar{N}_l}{EA}\xi + \bar{u}_0 - \frac{p_0}{EA}\left(\int_0^{\xi} (-x)\,dx + \int_{\xi}^{l} (-\xi)dx\right) \\
&= \frac{\bar{N}_l}{EA}\xi + \bar{u}_0 - \frac{p_0}{EA}\left(-\left[\frac{x^2}{2}\right]_0^{\xi} - \xi(l-\xi)\right) \\
&= \frac{\bar{N}_l}{EA}\xi + \bar{u}_0 - \frac{p_0}{EA}\left(-\frac{\xi^2}{2} - \xi(l-\xi)\right) \\
&= \frac{\bar{N}_l}{EA}\xi + \bar{u}_0 + \frac{p_0}{EA}\left(-\frac{\xi^2}{2} + \xi l\right) \qquad (3.102)
\end{aligned}$$

As easily can be checked, this is the exact solution of the problem.

Remark: The Green's functions for bar problems where

a) at $x = a$ a prescribed force $\bar{N}_0$ is acting and at $x = b$ the axial displacement is prescribed, i.e.,

$$N(x=a) = EA\left.\frac{du(x)}{dx}\right|_{x=a} = \bar{N}_0 \quad \text{and} \quad u(x=b) = \bar{u}_l \qquad (3.103)$$

is given by

$$G^{Nu}(x,\xi) = u^*(x,\xi) - u^*(b,\xi) - \left.\frac{\partial u^*(x,\xi)}{\partial x}\right|_{x=a}(x-b) \qquad (3.104)$$

b) at $x = a$ and at $x = b$ displacements are prescribed, i.e.,

$$u(x=a) = \bar{u}_0 \quad \text{and} \quad u(x=b) = \bar{u}_l \qquad (3.105)$$

is given by

$$G^{uu}(x,\xi) = u^*(x,\xi) - u^*(a,\xi)\frac{b-x}{b-a} - u^*(b,\xi)\frac{x-a}{b-a} \qquad (3.106)$$

3.1.3.2 Bending of beams

As given in (3.71), the direct form of the integral equation for the deflection of an elastic beam of length l is

$$\begin{aligned}
w(\xi) &= Q(l)w^*(l,\xi) - Q(0)w^*(0,\xi) - w(l)Q^*(l,\xi) + w(0)Q^*(0,\xi) \\
&\quad -M(l)w^{*'}(l,\xi) + w^{'}(l)M^*(l,\xi) + M(0)w^{*'}(0,\xi) - w^{'}(0)M^*(0,\xi) \\
&\quad + \int_0^l \bar{q}(x)\,w^*(x,\xi)dx \qquad (3.107)
\end{aligned}$$

where, the fundamental solution (see, (3.67))

$$w^*(x,\xi) = \frac{r^3}{12EI} \qquad (3.108)$$

satisfies the 1-dimensional Bi-potential equation

$$\frac{\partial^4 w^*(x,\xi)}{\partial x^4} = \frac{\delta(x,\xi)}{EI} \tag{3.109}$$

Considering as an actual problem

$$\frac{d^4 w(x)}{dx^4} = \frac{\bar{q}(x)}{EI} \tag{3.110}$$

with the boundary conditions

$$w(x = 0) = 0 \quad \text{and} \quad w'(x=0) = 0 \tag{3.111}$$
$$w(x = l) = \bar{w}_l \quad \text{and} \quad M(x=l) = 0 \tag{3.112}$$

i.e., a beam with a clamped boundary at $x = 0$ and with a pinned support at $x = l$ which has suffered a vertical settlement of $w(x = l) = \bar{w}_l$, the above integral equation reads as

$$\begin{aligned} w(\xi) &= Q(l)w^*(l,\xi) - Q(0)w^*(0,\xi) + w'(l)M^*(l,\xi) + M(0)w^{*'}(0,\xi) \\ &\quad -\bar{w}_l Q^*(l,\xi) + \int_0^l \bar{q}(x)\, w^*(x,\xi)dx \end{aligned} \tag{3.113}$$

The adequate Green's function $G^{cs}(x,\xi)$ of this problem has to satisfy the equation (3.109), i.e., has to be a fundamental solution, but has additionally to fulfill the homogeneous conditions

$$G^{cs}(l,\xi) = 0 \quad \text{and} \quad G^{cs}(0,\xi) = 0 \tag{3.114}$$
$$M^*(G^{cs}(x,\xi))\big|_{x=l} = -EI \left.\frac{\partial^2 G^{cs}(x,\xi)}{\partial x^2}\right|_{x=l} = 0 \tag{3.115}$$
$$\left.\frac{\partial G^{cs}(x,\xi)}{\partial x}\right|_{x=0} = 0 \tag{3.116}$$

Then, the deflection $w(\xi)$ at an arbitrary point solution ξ is simply

$$w(\xi) = -\bar{w}_l\, Q^*(G^{cs}(x,\xi))\big|_{x=l} + \int_0^l \bar{q}(x)\, G^{cs}(x,\xi)dx \tag{3.117}$$

The derivation of the Green's function $G^{cs}(x,\xi)$ of the above defined problem can start with an 'ansatz' which combines the fundamental solution $w^*(x,\xi)$ and its derivatives, respectively, with unknown polynomials $h_1(x), h_2(x), h_3(x)$, and $h_4(x)$ adequately

$$\begin{aligned} G^{cs}(x,\xi) &= w^*(x,\xi) - w^*(0,\xi)h_1(x) - w^*(l,\xi)h_2(x) \\ &\quad - \left.\frac{\partial w^*(x,\xi)}{\partial x}\right|_{x=0} h_3(x) - \left.\frac{\partial^2 w^*(x,\xi)}{\partial x^2}\right|_{x=l} h_4(x) \end{aligned} \tag{3.118}$$

These polynomials have to be cubic and, as follows from the conditions (3.114) to (3.116) must satisfy the conditions

$$\begin{aligned} h_1(0) &= 1, h_1(l)=0, h_1^{'}(0)=0, h_1^{''}(l)=0 \\ h_2(0) &= 0, h_2(l)=1, h_2^{'}(0)=0, h_2^{''}(l)=0 \\ h_3(0) &= 0, h_3(l)=0, h_3^{'}(0)=1, h_3^{''}(l)=0 \\ h_4(0) &= 0, h_4(l)=0, h_4^{'}(0)=0, h_4^{''}(l)=1 \end{aligned} \tag{3.119}$$

Some simple analysis gives

$$h_1(x) = 1-\frac{3}{2}\left(\frac{x}{l}\right)^2+\frac{1}{2}\left(\frac{x}{l}\right)^3 \tag{3.120}$$

$$h_2(x) = \frac{3}{2}\left(\frac{x}{l}\right)^2-\frac{1}{2}\left(\frac{x}{l}\right)^3 \tag{3.121}$$

$$h_3(x) = l\left[\left(\frac{x}{l}\right)-\frac{3}{2}\left(\frac{x}{l}\right)^2+\frac{1}{2}\left(\frac{x}{l}\right)^3\right] \tag{3.122}$$

$$h_3(x) = \frac{l^2}{4}\left[-\left(\frac{x}{l}\right)^2+\left(\frac{x}{l}\right)^3\right] \tag{3.123}$$

Then, introducing for $0 \le \xi \le l$ with $\partial r/\partial x = \text{sign}(x-\xi)$ and $r^2\delta(x-\xi)=0$

$$\begin{aligned} w^*(0,\xi) &= \frac{\xi^3}{12EI},\ w^*(l,\xi)=\frac{(l-\xi)^3}{12EI}, \\ \left.\frac{\partial w^*(x,\xi)}{\partial x}\right|_{x=0} &= -\frac{\xi^2}{4EI},\ \left.\frac{\partial^2 w^*(x,\xi)}{\partial x^2}\right|_{x=l}=\frac{(l-\xi)}{2EI} \end{aligned} \tag{3.124}$$

one obtains after some re-arrangements the Green's function explicitly as ($r^3=|x-\xi|^3=(x-\xi)^3\text{sign}(x-\xi)$)

$$G^{cs}(x,\xi)=\frac{1}{12EI}\left\{\begin{array}{c} r^3+3x\xi^2-\xi^3+3(\xi^3-3\xi^2 l+\xi l^2)\left(\frac{x}{l}\right)^2 \\ -(\xi^3-3\xi^2 l+l^3)\left(\frac{x}{l}\right)^3 \end{array}\right\} \tag{3.125}$$

Remark: It should be mentioned that the same result can be found by starting with the general polynomial 'ansatz'

$$G^{cs}(x,\xi)=\frac{1}{12EI}\left\{r^3+c_1x^3+c_2x^2+c_3x+c_4\right\} \tag{3.126}$$

and determing the four constants c_1, c_2, c_3, and c_4 via the four homogeneous boundary conditions (3.114) to (3.116).

Example: .By using the above Green's function, the deflection function $w(\xi)$ of a beam with a clamped boundary at $x=0$ and with a pinned support at $x=l$ which has a vertical settlement at $x=l$ of $w(x=l)=\bar{w}_l$ and is continuously loaded by $\bar{q}(x)=q_0$ is given as (see, (3.117))

$$w(\xi)=-\bar{w}_l\,Q^*(G^{cs}(x,\xi))|_{x=l}+q_0\int_0^l G^{cs}(x,\xi)dx$$

i.e., one has to evaluate at $x = l$ ($r_{,x}$ =sign$(x-\xi)$)

$$\begin{aligned} Q^*(G^{cs}(x,\xi)) &= -EI\frac{\partial^3}{x^3}G^{cs}(x,\xi) \\ &= -\frac{1}{12}\left\{6r_{,x} - (\xi^3 - 3\xi^2 l + l^3)\frac{6}{l^3}\right\}, \end{aligned}$$

i.e.

$$\begin{aligned} Q^*(G^{cs}(x,\xi))|_{x=l} &= -\frac{1}{2}\left\{1 - \left(\frac{\xi}{l}\right)^3 + 3\left(\frac{\xi}{l}\right)^2 - 1\right\} \\ &= \frac{1}{2}\left(\frac{\xi}{l}\right)^2\left\{\frac{\xi}{l} - 3\right\} \end{aligned}$$

and to integrate $G^{cs}(x,\xi)$ along the beam

$$\begin{aligned} \int_0^l G^{cs}(x,\xi)dx &= \frac{1}{12EI}\int_0^l \left\{ \begin{array}{c} r^3 + 3x\xi^2 - \xi^3 + 3(\xi^3 - 3\xi^2 l + \xi l^2)\left(\frac{x}{l}\right)^2 \\ -(\xi^3 - 3\xi^2 l + l^3)\left(\frac{x}{l}\right)^3 \end{array} \right\} dx \\ &= \frac{1}{12EI}\left\{ \begin{array}{c} \frac{1}{4}\left(\xi^4 + (l-\xi)^4\right) + \left[3\frac{x^2}{2}\xi^2 - x\xi^3\right]_0^l \\ +3(\xi^3 - 3\xi^2 l + \xi l^2)\left[\frac{l}{3}\left(\frac{x}{l}\right)^3\right]_0^l \\ -(\xi^3 - 3\xi^2 l + l^3)\left[\frac{l}{4}\left(\frac{x}{l}\right)^4\right]_0^l \end{array} \right\} \\ &= \frac{1}{12EI}\left\{ \begin{array}{c} \frac{1}{4}\left(\xi^4 + (l-\xi)^4\right) + \frac{3}{2}l^2\xi^2 - l\xi^3 \\ +(\xi^3 - 3\xi^2 l + \xi l^2)l - (\xi^3 - 3\xi^2 l + l^3)\frac{l}{4} \end{array} \right\} \\ &= \frac{1}{48EI}\left\{2\xi^4 - 5l\xi^3 + 3l^2\xi^2\right\} \end{aligned}$$

Remark: For $0 \le \xi \le l$, one has to integrate $r^3 = |x-\xi|^3$ as follows

$$\begin{aligned} \int_0^l r^3 dx &= \int_0^l |x-\xi|^3 dx \\ &= \int_0^\xi (\xi - x)^3 dx + \int_\xi^l (x-\xi)^3 dx \\ &= \left[-\frac{(\xi-x)^4}{4}\right]_0^\xi + \left[\frac{(x-\xi)^4}{4}\right]_\xi^l \\ &= \frac{1}{4}\left(\xi^4 + (l-\xi)^4\right) \end{aligned}$$

Finally, one finds the deflection for any position $\xi \in [0,l]$ to be

$$w(\xi) = -\bar{w}_l\frac{1}{2}\left(\frac{\xi}{l}\right)^2\left\{\frac{\xi}{l} - 3\right\} + \frac{q_0}{48EI}\left\{2\xi^4 - 5l\xi^3 + 3l^2\xi^2\right\}$$

which is, as easily can be checked, the exact solution for these boundary conditions and the constant loading.

3.1.3.3 Exercise 11: Green' functions for beam problems

Determine the adequate Green's function for a beam which has
a) a clamped support at both endings
b) a clamped support at $x = 0$ while the other ending is free:

3.1.4 Indirect integral formulations: the singularity method

One important feature of the *indirect method* is that the physical variables of the boundary value problems, the unknown boundary reactions, do not remain the unknown quantities of the integral equation: **Intermediary unknowns - unknown intensities of certain singularity layers - are introduced instead. For these singularities, so-called influence functions, i.e., the complete response to the action of a singularity (e.g., a unit point force) must be known everywhere in the considered material. Then, these singularity layers are distributed on a 'fictitious' boundary Γ^+ in a certain small distance d_ε from the real boundary Γ outside the domain Ω and their intensities have to be determined such that the integrated response is equal to the prescribed boundary values on the real boundary Γ.**

Since it is difficult to choose an optimal size of this small distance d_ε, it is the best choice to use $d_\varepsilon = 0$, i.e., let the fictitious boundary Γ^+ coincide (in the limit from outside) with the real boundary Γ.

3.1.4.1 Representation of Poisson equation problems: Stretching of bars

The differential equation for the axial displacement $u(x)$ of an elastic bar with sectional area A and modulus of elasticity E under the prescribed axial loading $p(x)$ is by (see (3.35), Example 6):

$$\frac{d^2u(x)}{dx^2} = -\frac{p(x)}{EA} \tag{3.127}$$

and we know (see (2.79)) the fundamental solution $u^*(x, \xi) = r/2$ of the Poisson equation

$$\frac{\partial^2 u^*(x, \xi)}{\partial x^2} = \delta(x, \xi)$$

Comparing both equations, it is obvious that from its physical meaning

$$(up)\,(x, \xi) = -\frac{u^*(x, \xi)}{EA} = -\frac{r}{2EA} \tag{3.128}$$

gives the axial displacement at the point x due to a axial unit point force at point ξ, i.e., is the influence function of a point force with intensity 1 for the axial displacement in the bar stretching problem.

Since the resultant axial force $N(x)$ is related to the axial displacement $u(x)$ via $N(x) = EAu^{'}(x)$, the corresponding influence function for this state is obtained by applying this definition to (3.128) as

$$(Np)(x,\xi) = EA\frac{\partial (up)(x,\xi)}{\partial x} = -\frac{1}{2}\frac{\partial r}{\partial x} = -\frac{1}{2}\left[2H(x-\xi)-1\right] = -\frac{1}{2}\mathrm{sgn}(x-\xi) \quad (3.129)$$

Following the above described idea of the indirect method, one has to introduce at the points ξ on the fictitious boundary Γ^+ (which is either enclosing the real boundary Γ with a certain distance d_ε or both boundaries coincide with each other) the intensity $p^*(\xi)$ of an adequate singularity, here, of a point forces, as new unknown function. Then, this intensity $p^*(\xi)$ of the singularity layers must take such a distribution that all prescribed boundary conditions on the real boundary Γ will be satisfied, either pointwise or in some other sense (certain norm).

For demonstrating this idea, the stretching of an elastic bar of length $l = b - a$ with sectional area A and Young's modulus E under the prescribed axial loading $\bar{p}(x)$ (see, Exercise 6) is considered with mixed boundary conditions: the bar shall be fixed at the boundary point $x = a$, i.e., $u(x = a) = 0$, and shall have a free ending at the other boundary point $x = b$, i.e., $N(x = b) = 0$.

The prescribed axial loading $\bar{p}$ has to be considered as a point force singularity layer with prescribed intensity $\bar{p}(\xi)$ in the interior of the bar's domain $\Omega = (a, b)$.

The axial displacement u at the point x caused by the point force distribution with unknown intensity $p^*(\xi)$ on the fictitiuos boundary $\Gamma^+$with an arbitrary small distance d_ε from the boundary points $x = a$ and $x = b$, i.e., at $\xi = a - d_\varepsilon$ and at $\xi = b + d_\varepsilon$, and by the axial loading in the bar's interior (a, b) with the prescribed intensity $\bar{p}(\xi)$ may be expressed applying the influence function $(up)(x,\xi)$ as

$$u(x) = \left[(up)(x,\xi)p^*(\xi)\right]_{\xi=a-d_\varepsilon}^{\xi=b+d_\varepsilon} + \int_a^b (up)(x,\xi)\,\bar{p}(\xi)d\xi \quad (3.130)$$

Correspondingly, using the influence function $(Np)(x,\xi)$, the resultant axial force $N(x)$ may be expressed as:

$$N(x) = \left[(Np)(x,\xi)p^*(\xi)\right]_{\xi=a-d_\varepsilon}^{\xi=b+d_\varepsilon} + \int_a^b (Np)(x,\xi)\bar{p}(\xi)d\xi \quad (3.131)$$

These two boundary 'integral' equations give the boundary value problem solution at all points x in the closed domain $a \le x \le b$ if the intensities $p^*(a - d_\varepsilon)$ and $p^*(b + d_\varepsilon)$ are determined so that the prescribed boundary conditions are satisfied, i.e.:

$$\bar{u}(a) = 0 = \left[(up)(a,\xi)p^*(\xi)\right]_{\xi=a-d_\varepsilon}^{\xi=b+d_\varepsilon} + \int_a^b (up)(a,\xi)\bar{p}(\xi)d\xi \quad (3.132)$$

$$\bar{N}(b) = 0 = [(Np)(b,\xi)p^*(\xi)]_{\xi=a-d_\varepsilon}^{\xi=b+d_\varepsilon} + \int_a^b (Np)(b,\xi)\bar{p}(\xi)d\xi \tag{3.133}$$

or, more detailed, by inserting the above defined influence functions ($r = |x-\xi|$, $l = b-a$, $d_\varepsilon \geqslant 0$)

$$\begin{aligned} -\frac{|a-b-d_\varepsilon|}{2EA}p^*(b+d_\varepsilon) + \frac{|a-a+d_\varepsilon|}{2EA}p^*(a-d_\varepsilon) &= \int_a^b \frac{|a-\xi|}{2EA}\bar{p}(\xi)d\xi \\ -(l+d_\varepsilon)\,p^*(b+d_\varepsilon) + d_\varepsilon\,p^*(a-d_\varepsilon) &= \int_a^b (\xi-a)\bar{p}(\xi)d\xi \end{aligned} \tag{3.134}$$

$$\begin{aligned} -\frac{\operatorname{sgn}(b-b-d_\varepsilon)}{2}p^*(b+d_\varepsilon) + \frac{\operatorname{sgn}(b-a+d_\varepsilon)}{2}p^*(a-d_\varepsilon) &= \int_a^b \frac{\operatorname{sgn}(b-\xi)}{2}\bar{p}(\xi)d\xi \\ p^*(b+d_\varepsilon) + p^*(a-d_\varepsilon) &= \int_a^b \bar{p}(\xi)d\xi \end{aligned} \tag{3.135}$$

For demonstrating the correctness of these two indirect boundary integral forms, an example shall be solved explicitly and compared with the exact analytical solution.

3.1.4.1.1 Example: Fixed-free bar under linear axial loading The above equations (3.134) and (3.135) are for a prescribed axial loading $\bar{p}(x) = p_0\frac{x-a}{l}$:

$$-(l+d_\varepsilon)\,p^*(b+d_\varepsilon) + d_\varepsilon\,p^*(a-d_\varepsilon) = \int_a^b (\xi-a)p_0\frac{\xi-a}{l}d\xi = \frac{p_0 l^2}{3} \tag{3.136}$$

and

$$p^*(b+d_\varepsilon) + p^*(a-d_\varepsilon) = \int_a^b p_0\frac{\xi-a}{l}d\xi = p_0\frac{l}{2} \tag{3.137}$$

These two equations give the sought intensities

$$p^*(a-d_\varepsilon) = \frac{p_0 l}{6}\frac{5l+3d_\varepsilon}{l+2d_\varepsilon} \;\text{ and }\; p^*(b+d_\varepsilon) = -\frac{p_0 l}{6}\frac{2l-3d_\varepsilon}{l+2d_\varepsilon} \tag{3.138}$$

For the special case $d_\varepsilon = 0$, i.e., for choosing the fictitious boundary to be identic with the real boundary, i.e., for $\Gamma^+ = \Gamma$, the result is simplified to

$$p^*(a) = \frac{5}{6}p_0 l \;\text{ and }\; p^*(b) = -\frac{1}{3}p_0 l \tag{3.139}$$

With the intensities (3.138) the indirect integral equations (3.130) can be evaluated for any arbitrary interior point x as

$$
\begin{aligned}
u(x) &= \left[-\frac{|x-\xi|}{2EA}p^*(\xi)\right]_{\xi=a-d_\varepsilon}^{\xi=b+d_\varepsilon} - \int_a^b \frac{|x-\xi|}{2EA}p_0\frac{\xi-a}{l}d\xi \\
&= \frac{p_0 l}{12EA}\left((b+d_\varepsilon-x)\frac{2l-3d_\varepsilon}{l+2d_\varepsilon}+(x-a+d_\varepsilon)\frac{5l+3d_\varepsilon}{l+2d_\varepsilon}\right) \\
&\quad -\frac{1}{2EA}\frac{p_0}{l}\left(\int_a^x - \int_x^b\right)(x-\xi)(\xi-a)d\xi \\
&= \frac{p_0 l}{12EA(l+2d_\varepsilon)}\left(3xl+2bl-5al+d_\varepsilon(6x-3a-3b+7l)\right) \\
&\quad -\frac{p_0}{2EAl}\left(\frac{(x-a)^3}{3}-\frac{(x-a)l^2}{2}+\frac{l^3}{3}\right) \qquad (3.140)
\end{aligned}
$$

At first, this solution does not look like the exact solution which easily can be determined by direct integrations to be

$$
u_{\text{exact}}(x) = -\frac{p_0}{EA}\frac{(x-a)}{6l}\left[(x-a)^2-3l^2\right] \qquad (3.141)
$$

but eliminating b in (3.140) by $b=a+l$ shows that one can cancel the term $(l+2d_\varepsilon)$ in

$$
\begin{aligned}
\frac{p_0 l\left(3xl+2bl-5al+d_\varepsilon(6x-3a-3b+7l)\right)}{12EA(l+2d_\varepsilon)} &= \frac{p_0 l\left(3l(x-a)+2l^2+d_\varepsilon(6(x-a)+4l)\right)}{12EA(l+2d_\varepsilon)} \\
&= \frac{p_0 l}{12EA(l+2d_\varepsilon)}\left(3(x-a)+2l\right)(l+2d_\varepsilon) \\
&= \frac{p_0 l}{12EA}\left(3(x-a)+2l\right) \qquad (3.142)
\end{aligned}
$$

Obviously , this term, i.e., the result of the boundary 'integral' on the fictitious boundary Γ^+, is independent on the distance d_ε. Now, subtracting from (3.142) the result of the domain integral

$$
\begin{aligned}
u(x) &= \frac{p_0 l}{12EA}\left(3(x-a)+2l\right) - \frac{p_0}{2EAl}\left(\frac{(x-a)^3}{3}-\frac{(x-a)l^2}{2}+\frac{l^3}{3}\right) \\
&= \frac{p_0}{12EAl}\left(3l^2(x-a)+2l^3-2(x-a)^3+3(x-a)l^2-2l^3\right) \\
&= \frac{p_0(x-a)}{6EAl}\left(3l^2-(x-a)^2\right)
\end{aligned}
$$

the exact solution (3.141) is obtained. Evaluating in the same way the indirect integral equation

(3.131) for the resultant axial force

$$\begin{aligned}
N(x) &= \left[-\frac{1}{2}\operatorname{sgn}(x-\xi)p^*(\xi)\right]_{\xi=a-d_\varepsilon}^{\xi=b+d_\varepsilon} - \int_a^b \frac{1}{2}\operatorname{sgn}(x-\xi)p_0\frac{\xi-a}{l}d\xi \\
&= \left[-\frac{1}{2}\frac{p_0 l}{6}\frac{2l-3d_\varepsilon}{l+2d_\varepsilon} + \frac{1}{2}\frac{p_0 l}{6}\frac{5l+3d_\varepsilon}{l+2d_\varepsilon}\right] - \frac{p_0}{2l}\left(\int_a^x - \int_x^b\right)(\xi-a)d\xi \\
&= \frac{p_0 l}{12}\frac{3l+6d_\varepsilon}{l+2d_\varepsilon} - \frac{p_0}{2l}\left((x-a)^2 - \frac{l^2}{2}\right) = \frac{p_0 l}{12}3 - \frac{p_0}{2l}\left((x-a)^2 - \frac{l^2}{2}\right) \\
&= \frac{p_0}{2l}\left(l^2 - (x-a)^2\right) \qquad (3.143)
\end{aligned}$$

gives again the exact solution.

Remark: Obviously, for one-dimensional problems where no approximation errors (no discretization of the boundary, no ansatz functions for the state functions, no numerical integration, no point collocation, and no numerical equation solver) occur, the exact solution is always obtained and the distance d_ε of the fictitious boundary Γ^+ does not influence the solution.

3.2 2-d and 3-d problems: Transformation of partial differential equations

In the case of two- and three-dimensional problems, a direct analytical integration of the partial differential equations is not possible. It is necessary to transform the differential equation either by the method of weighted residuals or by the so-called singularity method in integral equations. Their solution is generally possible only by using discretization techniques.

3.2.1 Direct integral equations by the method of weighted residuals

The Poisson or Laplace equation is a partial differential equation for a scalar function and can, therefore, be handled relatively simple. Hence, the transformation of a partial differential equation to an integral equation shall be demonstrated for this case first. Problems governed by this equation appear in different fields. Without aiming to be exhaustive a certain number of them have been summarized in the following table.

Problems $\Delta\Phi = 0$	Scalar function Φ	Boundary $\Phi = \bar{\Phi}$	conditions $\bar{q} = K\frac{\partial\Phi}{\partial n}$	Constants (K)
Heat transfer	Temperature $(T \equiv Deg.)$	$T = \bar{T}$	Heat flow $\bar{q} = -\lambda\frac{\partial T}{\partial n}$	Thermal conductivity (λ)
Ideal fluid flow	Stream function $(\Phi \equiv m^2 s^{-1})$	$\Phi = \bar{\Phi}$	$\bar{q} = \frac{\partial\Phi}{\partial n}$	
Groundwater flow	Hydraulic head $(\Phi \equiv m)$	$\Phi = \bar{\Phi}$	$\bar{q} = K\frac{\partial\Phi}{\partial n}$	Permeability (K)
Hydrodynamic pressure on moving surfaces	Pressure $(P \equiv Nm^{-2})$	$P = \bar{P}$ free surface: $P = 0$	$\bar{q} = -\rho a_n$ (a_n normal accel.)	Density (ρ)
Electrostatic	Field potential $(V \equiv \text{volt})$	$V = \bar{V}$	$\bar{q} = -\varepsilon\frac{\partial V}{\partial n}$	Permittivity (ε)
Electric conduction	Electropotential $(E \equiv \text{volt})$	$E = \bar{E}$	$\bar{q} = \frac{1}{k}\frac{\partial E}{\partial n}$	Resistivity (k)
Magnetostatic	Magnetic Potential $(M \equiv \text{amp})$	$M = \bar{M}$	$\bar{B} = -\mu\frac{\partial M}{\partial n}$	Magnetic Permeability (μ)

3.2.1.1 Stationary heat conduction - a Poisson equation

In the case of a homogeneous isotropic body with constant thermal conductivity λ_0, the stationary (not longer changing with time) temperature field $\Theta(\mathbf{x})$ is governed by the scalar Poisson equation

$$\Delta\Theta(\mathbf{x}) = -\frac{1}{\lambda_0} W_q(\mathbf{x}) \tag{3.144}$$

where $W_q(\mathbf{x})$ represents the heat source generation rate. Associated boundary conditions may involve either a fixed given temperature (a so-called Dirichlet boundary condition) at a part of the boundary Γ

$$\Theta(\mathbf{x}) = \bar{\Theta}(\mathbf{x}) \;\; \text{for } \mathbf{x} \in \Gamma \tag{3.145}$$

or a prescribed heat flux through a boundary part (a so-called Neumann boundary condition):

$$q_n(\mathbf{x}) = \lambda_0 \frac{\partial\Theta(\mathbf{x})}{\partial n(\mathbf{x})} = \bar{q}_n(\mathbf{x}) \;\; \text{für } \mathbf{x} \in \Gamma \tag{3.146}$$

where $\mathbf{n}(\mathbf{x})$ is the outward normal unit vector. This heat flux can be zero when the boundary is insulated.

Almost no real problems have purely temperature or flux specified boundary conditions, so it is necessary to consider mixed boundary conditions from the beginning, i.e., one has on one boundary part Γ_1 the condition (3.145) and on the remaining part Γ_2 the condition (3.146).

Now, as in the above discussed one-dimensional problems, the weighted residual of the considered differential equation (3.144) is integrated over the domain Ω of the problem and set to be zero:

$$\int_{\Omega} \left[\Delta\Theta(\mathbf{x}) + \frac{W_q(\mathbf{x})}{\lambda_0} \right] \Theta^*(\mathbf{x},\xi) d\Omega_{\mathbf{x}} = 0 \tag{3.147}$$

where the fundamental solution $\Theta^*(\mathbf{x},\xi)$ of the respective differential equation is taken as special weighting function.

The respective fundamental solutions are known (see, (2.88) and (2.86)) to be ($r =| \mathbf{x} - \xi |$):

$$\Theta^*(\mathbf{x},\xi) = \frac{-1}{2\pi} \ln(\frac{r}{c}) = \frac{-1}{2\pi}[\ln(r) - \ln(c)] \ \text{ in } R^2 \tag{3.148}$$

$$\Theta^*(\mathbf{x},\xi) = \Theta^*(r) = \frac{1}{4\pi r} \ \text{ in } R^3 \tag{3.149}$$

where $c > 0$ is an arbitrary real constant making the ratio r/a dimensionless, e.g., in the case of a numerical solution procedure, taken as the smallest geometrical dimension of the discretization.

Now, following the above introduced rules for deriving an equivalent integral equation representation by the method of weighted residuals, the first differential operator term in (3.147) has to integrated by parts till all differentiations are transfered from the unknown function $\Theta(\mathbf{x})$ to the known weighting function, the fundamental solution $\Theta^*(\mathbf{x},\xi)$. This gives ($\frac{\partial}{\partial x_i} =_{,i}$) for $\xi \in \Omega$:

$$\begin{aligned}
\int_{\Omega} \Delta\Theta(\mathbf{x})\Theta^*(\mathbf{x},\xi) d\Omega_{\mathbf{x}} &= \int_{\Omega} [\Theta(\mathbf{x})]_{,ii}\Theta^*(\mathbf{x},\xi) d\Omega_{\mathbf{x}} \\
&= \int_{\Omega} \Big(\{[\Theta(\mathbf{x})]_{,i}\Theta^*(\mathbf{x},\xi)\}_{,i} - [\Theta(\mathbf{x})]_{,i}[\Theta^*(\mathbf{x},\xi)]_{,i} \Big) d\Omega_{\mathbf{x}} \\
&= \int_{\Gamma} [\Theta(\mathbf{x})]_{,i}\Theta^*(\mathbf{x},\xi) n_i(\mathbf{x}) d\Gamma_{\mathbf{x}} - \int_{\Omega} [\Theta(\mathbf{x})]_{,i}[\Theta^*(\mathbf{x},\xi)]_{,i} d\Omega_{\mathbf{x}} \\
&= \int_{\Gamma} \frac{1}{\lambda_0} q_n(\mathbf{x})\Theta^*(\mathbf{x},\xi) d\Gamma_{\mathbf{x}} \\
&\quad - \int_{\Omega} \Big(\{\Theta(\mathbf{x})[\Theta^*(\mathbf{x},\xi)]_{,i}\}_{,i} - \Theta(\mathbf{x})[\Theta^*(\mathbf{x},\xi)]_{,ii} \Big) d\Omega_{\mathbf{x}} \\
&= \int_{\Gamma} \frac{1}{\lambda_0} q_n(\mathbf{x})\Theta^*(\mathbf{x},\xi) d\Gamma_{\mathbf{x}} - \int_{\Gamma} \Theta(\mathbf{x})[\Theta^*(\mathbf{x},\xi)]_{,i} n_i(\mathbf{x}) d\Gamma_{\mathbf{x}} \\
&\quad + \int_{\Omega} \Theta(\mathbf{x})\delta(\mathbf{x},\xi) d\Omega_{\mathbf{x}} \\
&= \int_{\Gamma} \frac{1}{\lambda_0} \left[q_n(\mathbf{x})\Theta^*(\mathbf{x},\xi) - \Theta(\mathbf{x}) q_n^*(\mathbf{x},\xi) \right] d\Gamma_{\mathbf{x}} - \Theta(\xi)
\end{aligned} \tag{3.150}$$

where, as defined in (3.146), the heat flux $q_n(\mathbf{x})$ was introduced as second boundary state. Combining this transformed expression with the other terms of (3.147) delivers the following integral equation for the temperature at an arbitrary interior point $\xi \in \Omega$:

$$\Theta(\xi) = \int_\Gamma \frac{1}{\lambda_0} \left(q_n(\mathbf{x})\Theta^*(\mathbf{x},\xi) - \Theta(\mathbf{x}) q_n^*(\mathbf{x},\xi) \right) d\Gamma_{\mathbf{x}} + \int_\Omega \frac{W_q(\mathbf{x})}{\lambda_0} \Theta^*(\mathbf{x},\xi) d\Omega_{\mathbf{x}} \tag{3.151}$$

where

$$\begin{aligned} q_n^*(\mathbf{x},\xi) &= \lambda_0 [\Theta^*(\mathbf{x},\xi)]_{,i} n_i(\mathbf{x}) \\ &= \frac{-\lambda_0}{2\pi} \frac{1}{r} \frac{\partial r}{\partial x_i} n_i(\mathbf{x}) = \frac{-\lambda_0}{2\pi} \frac{(x_i - \xi_i)}{r^2} n_i(\mathbf{x}) \text{ in } R^2 && (3.152) \\ &= \frac{-\lambda_0}{4\pi r^2} \frac{\partial r}{\partial x_i} n_i(\mathbf{x}) = \frac{-\lambda_0}{4\pi} \frac{(x_i - \xi_i)}{r^3} n_i(\mathbf{x}) \text{ in } R^3 && (3.153) \end{aligned}$$

This integral equation (3.151) contains unknown boundary terms: the heat flux, where the temperature is prescribed, and the temperature, where the flux is prescribed. In order to obtain equations which are only dependent on boundary values and can be used to determine the unknown boundary reactions, the point ξ has to be shifted from the interior to the boundary Γ.

Then, the kernel $\Theta^*(\mathbf{x},\xi)$ and $q_n^*(\mathbf{x},\xi)$ in the boundary integral becomes weakly and strongly singular, respectively, when integration points $\mathbf{x}$ coincide with ξ, and, hence, it is necessary to avoid this. For this purpose, in the two-(three-)dimensional case, a small ε-intervall Γ_ε ahead and behind (circular region of radius ε around) ξ is cut out on the boundary line (surface) Γ, and the integration around ξ is performed along a circle line (on a spherical surface) Γ_ε^* with radius ε (with $\lim \varepsilon \to 0$):

$$\Theta(\xi) = \int_{\Gamma-\Gamma_\varepsilon+\Gamma_\varepsilon^*} \frac{1}{\lambda_0} \left[q_n(\mathbf{x})\Theta^*(\mathbf{x},\xi) - \Theta(\mathbf{x}) q_n^*(\mathbf{x},\xi) \right] d\Gamma_{\mathbf{x}} + \int_{\Omega'} \frac{W_q(\mathbf{x})}{\lambda_0} \Theta^*(\mathbf{x},\xi) d\Omega_{\mathbf{x}} \tag{3.154}$$

where $\Omega' = \Omega - \Omega_\varepsilon$ with $\Omega_\varepsilon = \{\mathbf{x} \in \Omega : | \mathbf{x} - \xi | \leq \varepsilon\}$.

The integrals with the weakly singular kernel $\Theta^*(\mathbf{x},\xi)$ can be evaluated as improper integrals, while those with the strongly singular kernel $q_n^*(\mathbf{x},\xi)$ can be determined on $\Gamma - \Gamma_\varepsilon$ as Cauchy principal values (see, above the respective section), and on Γ_ε^*, one has to consider two integrals

$$\begin{aligned} \int_{\Gamma-\Gamma_\varepsilon+\Gamma_\varepsilon^*} \frac{1}{\lambda_0} \Theta(\mathbf{x}) q_n^*(\mathbf{x},\xi) d\Gamma_{\mathbf{x}} &= \int_{\Gamma-\Gamma_\varepsilon} \frac{1}{\lambda_0} \Theta(\mathbf{x}) q_n^*(\mathbf{x},\xi) d\Gamma_{\mathbf{x}} + \Theta(\xi) \frac{1}{\lambda_0} \int_{\Gamma_\varepsilon^*} q_n^*(\mathbf{x},\xi) d\Gamma_{\mathbf{x}} \\ &\quad + \int_{\Gamma_\varepsilon^*} \frac{1}{\lambda_0} [\Theta(\mathbf{x}) - \Theta(\xi)] q_n^*(\mathbf{x},\xi) d\Gamma_{\mathbf{x}} && (3.155) \end{aligned}$$

where the second exists as an improper integral delivering zero since the temperature field is continous while the first one can directly be evaluated. One obtains in R^2 with

$\partial r/\partial n = 1$ and $d\Gamma_{\mathbf{x}} = r d\varphi$ and in R^3 with $n_i(\mathbf{x}) d\Gamma_{\mathbf{x}} = \mathbf{r}_{,\varphi} \times \mathbf{r}_{,\theta} d\varphi d\theta$ (see [1], p.37)

$$\frac{1}{\lambda_0} \int_{\Gamma_\varepsilon^*} q_n^*(\mathbf{x},\xi) d\Gamma_{\mathbf{x}} = \frac{-1}{2\pi} \int_{\varphi_1}^{\varphi_2} \frac{1}{r} \frac{\partial r}{\partial n} r d\varphi = -\frac{\varphi_2 - \varphi_1}{2\pi} \text{ in } R^2 \tag{3.156}$$

$$\frac{1}{\lambda_0} \int_{\Gamma_\varepsilon^*} q_n^*(\mathbf{x},\xi) d\Gamma_{\mathbf{x}} = \frac{-1}{4\pi} \int_{\Gamma_\varepsilon^*} \frac{(x_i - \xi_i)}{r^3} n_i(\mathbf{x}) d\Gamma_{\mathbf{x}} = \frac{-1}{4\pi} \int \int \sin\varphi d\varphi d\theta \text{ in } R^3 \tag{3.157}$$

where, in R^2, $\varphi_2 - \varphi_1$ means the 'external angle' of the boundary Γ at the point ξ, i.e., the difference of the outer normal direction at the beginning and at the end of Γ_ε^*. Finally, one obtains from (3.154) the following boundary integral equation:

$$\frac{\Delta\Omega(\xi)}{2\pi} \Theta(\xi) = \int_{\Gamma - \Gamma_\varepsilon} \frac{1}{\lambda_0} \left[q_n(\mathbf{x}) \Theta^*(\mathbf{x},\xi) - \Theta(\mathbf{x}) q_n^*(\mathbf{x},\xi) \right] d\Gamma_{\mathbf{x}} + \int_{\Omega'} \frac{W_q(\mathbf{x})}{\lambda_0} \Theta^*(\mathbf{x},\xi) d\Omega_{\mathbf{x}} \tag{3.158}$$

where for Ω in R^2, $\Delta\Omega(\xi) = 2\pi - (\varphi_2 - \varphi_1)$ means the internal angle of Γ at the point ξ, i.e., $\Delta\Omega(\xi) = \pi$ for all points ξ besides for corner points while for Ω in R^3, $\Delta\Omega(\xi)$ means the inner solid angle, i.e., $\Delta\Omega(\xi) = 2\pi$ for all points ξ besides for points at corners and edges.

3.2.1.2 Stationary sound radiation - a Helmholtz equation

Adding to (or subtracting from) the Laplace operator Δ a constant factor κ^2 gives the so-called Helmholtz operator, e.g., for the stationary sound radiation problem

$$\Delta p(\mathbf{x}) + \kappa^2 p(\mathbf{x}) = -b(\mathbf{x}) = +\mathrm{i} c \kappa \rho_0 a(\mathbf{x}) \tag{3.159}$$

where $p(\mathbf{x})$ is the sound pressure distribution when considering time-harmonic processes, the so-called wave number $\kappa = \omega/c$ with the excitation frequency ω and the sound speed c in the considered medium (air, water, a.s.o) with the density ρ_0, and $a(\mathbf{x})$ is the sound source intensity distribution.

Remark: In general, the Helmholtz equation is the result of a resolution in the Fourier domain or in the Laplace domain of a transient dynamical problem, or is decribing the response to steady-state excitations assuming that a permanent regime has been reached. In any case, the field variables are time-harmonic with a fixed angular frequency ω, i.e., of the form

$$p(\mathbf{x},t) = \Re[\hat{p}(\mathbf{x}) \mathrm{e}^{-\mathrm{i}\omega t}] \tag{3.160}$$

where $\hat{p}(\mathbf{x})$ is a complex-valued function which encodes amplitude and phase information. In the sequel, following the traditional convention, the factor $\mathrm{e}^{\mathrm{i}\omega t}$ is systematically omitted and the notation $p(\mathbf{x})$ is used instead of $\hat{p}(\mathbf{x})$.

Associated boundary conditions may involve either a fixed given sound pressure (a so-called Dirichlet boundary condition) at a part of the boundary Γ

$$p(\mathbf{x}) = \bar{p}(\mathbf{x}) \text{ for } \mathbf{x} \in \Gamma_1 \tag{3.161}$$

or a prescribed sound flux through a boundary part (a so-called Neumann boundary condition):

$$q(\mathbf{x}) = \frac{\partial p(\mathbf{x})}{\partial n(\mathbf{x})} = \bar{q}(\mathbf{x}) \;\; \text{für } \mathbf{x} \in \Gamma_2 \tag{3.162}$$

where $\mathbf{n}(\mathbf{x})$ is the outward normal unit vector, or a certain connexion between both can be described.

The boundary condition $p = 0$ models a 'free' surface, e.g., the free surface of a water domain when gravity waves are neglected, while $q = 0$ describes the complete reflexion of an incoming pressure wave. A boundary condition $q(\mathbf{x}) = \bar{q}(\mathbf{x}) \neq 0$ means a 'sound production' with a prescribed spatial change of intensity.

A partial reflexion, i.e., a damped reflexion of waves may be described by ($\mathrm{i}^2 = -1$)

$$q(\mathbf{x}) = \mathrm{i}\omega A p(\mathbf{x}) = \mathrm{i}\omega \frac{1-\alpha}{1+\alpha}\frac{1}{c} p(\mathbf{x}) \tag{3.163}$$

where the damping coefficient A depends on the reflexion coefficient $-1 \leq \alpha \leq 1$, who describes the ratio of reflected to incoming pressure wave, i.e., $\alpha = 0$ means no reflexion and $\mid \alpha \mid = 1$ a complete reflexion, either symmetric or antimetric corresponding to the sign of α.

The weighed residual of the Helmholtz equation (3.159) is similar to that of the Poisson equation (3.147)

$$\int_\Omega \left[\Delta p(\mathbf{x}) + \kappa^2 p(\mathbf{x}) + b(\mathbf{x})\right] p^*(\mathbf{x}, \xi) d\Omega_{\mathbf{x}} = 0 \tag{3.164}$$

The adequate fundamental soution $p^*(\mathbf{x}, \xi)$ is for a real κ (see (2.90) and (2.91))

$$p^*(\mathbf{x}, \xi) = p^*(r) = \frac{1}{4\pi r} \mathrm{e}^{-i\kappa r} \;\; \text{in } R^3 \tag{3.165}$$

$$p^*(\mathbf{x}, \xi) = p^*(r) = -\frac{i}{4} H_0^{(2)}(\kappa r) = \frac{1}{2\pi} K_0(\mathrm{i}\kappa r) \;\; \text{in } R^2 \tag{3.166}$$

where $H_0^{(2)}(kr)$ is a Hankel function of second kind and order zero, while $K_0(\mathrm{i}\kappa r)$ is a modified Besselfunction of order zero (Macdonald function).

Remarks: The derivatives of this Besselfunction are obtained by the following rules ($r_{,k} = \partial r / \partial x_k$):

$$\frac{\partial K_0(\mathrm{i}\kappa r)}{\partial x_k} = -\mathrm{i}\kappa K_1(\mathrm{i}\kappa r) \frac{\partial r}{\partial x_k} \tag{3.167}$$

$$\frac{\partial^2 K_0(\mathrm{i}\kappa r)}{\partial x_k \partial x_j} = -\kappa^2 K_0(\mathrm{i}\kappa r) r_{,k} r_{,j} + \frac{\mathrm{i}\kappa}{r} K_1(\mathrm{i}\kappa r) \left\{2 r_{,k} r_{,j} - \delta_{kj}\right\} \tag{3.168}$$

such that with $(r_{,1})^2 + (r_{,2})^2 = 1$ and $\delta_{11} + \delta_{22} = 2$ the homogeneous Helmholtz equation is shown to be satisfied for $\mathbf{x} \neq \xi$:

$$\Delta K_0(\mathrm{i}\kappa r) = \frac{\partial^2 K_0(\mathrm{i}\kappa r)}{\partial x_1^2} + \frac{\partial^2 K_0(\mathrm{i}\kappa r)}{\partial x_2^2} = -\kappa^2 K_0(\mathrm{i}\kappa r)$$

For (3.168), the following recursion formula have been used:

$$\begin{aligned} -2\frac{d}{dy}K_n(y) &= K_{n-1}(y) + K_{n+1}(y) \\ \frac{-2n}{y}K_n(y) &= K_{n-1}(y) - K_{n+1}(y) \end{aligned}$$

which gives by eliminating $K_{n+1}(y)$

$$\begin{aligned} \frac{d}{dy}K_n(y) &= -K_{n-1}(y) - \frac{n}{y}K_n(y), \\ \text{i.e., } \frac{d}{dy}K_1(y) &= -(K_0(y) + \frac{1}{y}K_1(y)) \end{aligned}$$

Hence, one obtains with $y = \mathrm{i}\kappa r$

$$\begin{aligned} \frac{\partial}{\partial x_k}K_1(\mathrm{i}\kappa r) &= \frac{d}{dy}K_1(y)\frac{\partial y}{\partial x_k} \\ &= -(K_0(y) - \frac{1}{y}K_1(y))\mathrm{i}\kappa r_{,k} \\ &= -(\mathrm{i}\kappa K_0(\mathrm{i}\kappa r) - \frac{1}{r}K_1(\mathrm{i}\kappa r))r_{,k} \end{aligned}$$

The integration by parts of the first integral term in (3.164) is formally identic to that in the case of the Poisson equation in (3.150) and can, therefore, be transfered (only the constant factor λ_0 has to be taken as 1). Hence, one obtains the following integral transformation of (3.164)

$$\begin{aligned} \int_\Omega \left[\Delta p(\mathbf{x}) + \kappa^2 p(\mathbf{x}) + b(\mathbf{x})\right] p^*(\mathbf{x},\xi) d\Omega_\mathbf{x} &= \int_\Gamma \left[q_n(\mathbf{x})p^*(\mathbf{x},\xi) - p(\mathbf{x})q_n^*(\mathbf{x},\xi)\right] d\Gamma_\mathbf{x} \\ &+ \int_\Omega p(\mathbf{x}) \left[\Delta p^*(\mathbf{x},\xi) + \kappa^2 p^*(\mathbf{x},\xi)\right] d\Omega_\mathbf{x} \\ &+ \int_\Omega b(\mathbf{x}) p^*(\mathbf{x},\xi) d\Omega_\mathbf{x} \end{aligned} \tag{3.169}$$

or, since $\Delta p^*(\mathbf{x},\xi) + \kappa^2 p^*(\mathbf{x},\xi) = -\delta(\mathbf{x},\xi)$ due to the filtering effect of the δ-function

$$p(\xi) = \int_\Gamma \left[q_n(\mathbf{x})p^*(\mathbf{x},\xi) - p(\mathbf{x})q_n^*(\mathbf{x},\xi)\right] d\Gamma_\mathbf{x} + \int_\Omega b(\mathbf{x})\, p^*(\mathbf{x},\xi) d\Omega_\mathbf{x} \tag{3.170}$$

where the normal derivative of the fundamental solution $q_n^*(\mathbf{x},\xi)$ is (see, for R^2, (3.167))

$$q_n^*(\mathbf{x},\xi) = \frac{\partial p^*(\mathbf{x},\xi)}{\partial x_k} n_k(\mathbf{x}) = -\frac{r_{,k}}{4\pi r^2}\left[1 + \mathrm{i}\kappa r\right] \mathrm{e}^{-i\kappa r} \text{ in } R^3 \tag{3.171}$$

$$q_n^*(\mathbf{x},\xi) = \frac{\partial p^*(\mathbf{x},\xi)}{\partial x_k} n_k(\mathbf{x}) = -\frac{\mathrm{i}\kappa}{2\pi} K_1(\mathrm{i}\kappa r) r_{,k} n_k(\mathbf{x}) \text{ in } R^2 \tag{3.172}$$

Since the integral equation (3.170) for the sound pressure at interior points $\xi \in \Omega$ contains unknown boundary reaction terms, one needs a boundary integral equation for their determination. Hence, as for the Poisson equation analysis, ξ has to be shifted on the boundary Γ, whereby the integral kernels $p^*(\mathbf{x},\xi)$ and $q_n^*(\mathbf{x},\xi)$ become weakly and strongly singular, respectively, for $\mathbf{x} \to \xi$ as well in R^3 due to the $1/r$ and $1/r^2$ behaviour as in R^2 since $\mathrm{K}_0(z) \to -\ln(z)$ and $\mathrm{K}_1(z) \to 1/z$.

As already explained in detail in the derivation of (3.158), the weakly singular integral in (3.170) exists as improper integral while the integral with the strongly singular kernel $q_n^*(\mathbf{x},\xi)$

$$\int\limits_{\Gamma-\Gamma_\varepsilon+\Gamma_\varepsilon^*} p(\mathbf{x})q_n^*(\mathbf{x},\xi)d\Gamma_{\mathbf{x}} = \int\limits_{\Gamma-\Gamma_\varepsilon} p(\mathbf{x})q_n^*(\mathbf{x},\xi)d\Gamma_{\mathbf{x}} + p(\xi)\int\limits_{\Gamma_\varepsilon^*} q_n^*(\mathbf{x},\xi)d\Gamma_{\mathbf{x}} \tag{3.173}$$

exists on $\Gamma - \Gamma_\varepsilon$ as Cauchy principal value, and can be evaluated on Γ_ε^* explicitly giving the same factors as in the Poisson equation case, e.g.:

$$\int\limits_{\Gamma_\varepsilon^*} q_n^*(\mathbf{x},\xi)d\Gamma_{\mathbf{x}} = -\int\limits_{\Gamma_\varepsilon^*} \frac{\mathrm{i}\kappa}{2\pi}K_1(\mathrm{i}\kappa r)\frac{\partial r}{\partial n}d\Gamma_{\mathbf{x}} = -\frac{\mathrm{i}\kappa}{2\pi}\int\limits_{\varphi_1}^{\varphi_2}\frac{1}{\mathrm{i}\kappa\varepsilon}\varepsilon d\varphi = -\frac{\varphi_2-\varphi_1}{2\pi} \quad \text{in } R^2$$

Finally, one obtains from (3.170) with $b(\mathbf{x}) = -ic\kappa\rho_0 a(\mathbf{x})$ the boundary integral equation

$$c(\xi)p(\xi) = \int\limits_{\Gamma-\Gamma_\varepsilon} \left[q_n(\mathbf{x})p^*(\mathbf{x},\xi) - p(\mathbf{x})q_n^*(\mathbf{x},\xi)\right] d\Gamma_{\mathbf{x}} - \int\limits_{\Omega'} ic\kappa\rho_0 a(\mathbf{x})\, p^*(\mathbf{x},\xi)d\Omega_{\mathbf{x}} \tag{3.174}$$

with

$$c(\xi) = \frac{\Delta\varphi(\xi)}{2\pi} = 1 - \frac{\varphi_2-\varphi_1}{2\pi} \quad \text{in } R^2$$

which is, besides the use of different fundamental solutions, formally almost identic to(3.158). The main difference to the scalar integral equation for the Poisson equation is the necessity of calculating with complex numbers.

3.2.1.3 Linear elastostatics - the Navier equations

When the state variables are vectorial states, boundary value problems are described by systems of partial differential equations. As a representative example, the Navier equations describing the deformation displacements $\mathbf{u}(\mathbf{x})$ of an linearly elastic body under body forces $\mathbf{b}(\mathbf{x})$ are considered here:

$$\mu\Delta\mathbf{u}(\mathbf{x}) + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u}(\mathbf{x}) = -\mathbf{b}(\mathbf{x}) \tag{3.175}$$

where the Lamé constants μ, ν are related to the Young's modulus E and the Poisson's ratio ν by $\mu = G = \frac{E}{2(1+\nu)}$ and $\lambda = \frac{\nu E}{(1-2\nu)(1+\nu)}$ for three-dimensional and plane strain states and $\lambda = \frac{\nu E}{1-\nu^2}$ for plane stress states, respectively.

The weighted residual for (3.175) is in indicial notation

$$\int\limits_{\Omega} \left[\mu\frac{\partial^2 u_i(\mathbf{x})}{\partial x_j \partial x_j} + (\lambda+\mu)\frac{\partial^2 u_j(\mathbf{x})}{\partial x_j \partial x_i} + b_i(\mathbf{x})\right] u_i^{(k)}(\mathbf{x},\xi)d\Omega_{\mathbf{x}} = 0 \tag{3.176}$$

where the adequate fundamental solution, the so-called Kelvin solution $u_i^{(k)}(\mathbf{x}, \xi)$ representing the response to a unit point force $b_i^*(\mathbf{x}) = \delta(\mathbf{x} - \xi)e_i^{(k)}$ applied at a given fixed point $\xi \in \Omega$ along the k-direction, is given by (note the different definitions of λ in R^2 and in R^3)

$$u_i^{(k)}(\mathbf{x}, \xi) = \frac{1}{4\pi}\frac{1}{2\mu + \lambda}\left[-(3 + \frac{\lambda}{\mu})\delta_{ik}\ln r + (1 + \frac{\lambda}{\mu})r_{,i}r_{,k}\right] \text{ in } R^2 \tag{3.177}$$

$$= \frac{1}{8\pi}\frac{1}{2\mu + \lambda}\frac{1}{r}\left[(3 + \frac{\lambda}{\mu})\delta_{ik} + (1 + \frac{\lambda}{\mu})r_{,i}r_{,k}\right] \text{ in } R^3 \tag{3.178}$$

For the integration by parts, it is helpful to substitute the Navier equations, which are the displacement representation of the interior equilibrium to the body forces, by its original stress-based form

$$\mu\frac{\partial^2 u_i(\mathbf{x})}{\partial x_j \partial x_j} + (\lambda + \mu)\frac{\partial^2 u_j(\mathbf{x})}{\partial x_j \partial x_i} = \frac{\partial \sigma_{ik}(\mathbf{x})}{\partial x_k} = -b_i(\mathbf{x}) \tag{3.179}$$

since, then, it is easy to perform the first integration by parts:

$$\begin{aligned}
\int_\Omega \frac{\partial \sigma_{ij}(\mathbf{x})}{\partial x_j} u_i^{(k)}(\mathbf{x}, \xi) d\Omega_{\mathbf{x}} &= \int_\Omega \left[[\sigma_{ij}(\mathbf{x})u_i^{(k)}(\mathbf{x}, \xi)]_{,j} - \sigma_{ij}(\mathbf{x})[u_i^{(k)}(\mathbf{x}, \xi)]_{,j}\right] d\Omega_{\mathbf{x}} \\
&= \int_\Gamma \sigma_{ij}(\mathbf{x})u_i^{(k)}(\mathbf{x}, \xi)n_j(\mathbf{x})d\Gamma_{\mathbf{x}} - \int_\Omega \sigma_{ij}(\mathbf{x})\frac{\partial u_i^{(k)}(\mathbf{x}, \xi)}{\partial x_j} d\Omega_{\mathbf{x}} \\
&= \int_\Gamma T_i(\mathbf{x})u_i^{(k)}(\mathbf{x}, \xi)d\Gamma_{\mathbf{x}} - \int_\Omega \sigma_{ij}(\mathbf{x})\frac{\partial u_i^{(k)}(\mathbf{x}, \xi)}{\partial x_j} d\Omega_{\mathbf{x}}
\end{aligned} \tag{3.180}$$

where $T_i = \sigma_{ij}n_j$ is the so-called traction vector.

From the definition of the strain tensor $\varepsilon_{ij} = 0.5(u_{i,j} + u_{j,i})$, one obtains due to the symmetry of the stress tensor σ_{ij}, the symmetry of the elastic constitutive law for isotropic material $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{ll}$, and the reciprocity with its inverse form $\varepsilon_{ij} = (\sigma_{ij} - \lambda\delta_{ij}\sigma_{ll}/(2\mu + 3\lambda))/2\mu$ and $\varepsilon_{ij} = (\sigma_{ij} - \lambda\delta_{ij}\sigma_{ll}/(2\mu + 2\lambda))/2\mu$ in 3D and 2D, respectively, i.e., $\sigma_{ij}^*\varepsilon_{ij} = \varepsilon_{ij}^*\sigma_{ij}$

$$\begin{aligned}
\sigma_{ij}(\mathbf{x})\frac{\partial u_i^{(k)}(\mathbf{x}, \xi)}{\partial x_j} &= \sigma_{ij}(\mathbf{x})\frac{1}{2}\left(\frac{\partial u_i^{(k)}(\mathbf{x}, \xi)}{\partial x_j} + \frac{\partial u_j^{(k)}(\mathbf{x}, \xi)}{\partial x_i}\right) = \sigma_{ij}(\mathbf{x})\varepsilon_{ij}^{(k)}(\mathbf{x}, \xi) \\
&= \varepsilon_{ij}(\mathbf{x})\sigma_{ij}^{(k)}(\mathbf{x}, \xi) = \frac{\partial u_i(\mathbf{x})}{\partial x_j}\sigma_{ij}^{(k)}(\mathbf{x}, \xi)
\end{aligned} \tag{3.181}$$

which allows the second integration by parts of the remaining domain integral

$$\begin{aligned}\int\limits_{\Omega} \frac{\partial u_i(\mathbf{x})}{\partial x_j}\sigma_{ij}^{(k)}(\mathbf{x},\xi)\,d\Omega_{\mathbf{x}} &= \int\limits_{\Omega} \Big[[u_i(\mathbf{x})\sigma_{ij}^{(k)}(\mathbf{x},\xi)]_{,j} - u_i(\mathbf{x})[\sigma_{ij}^{(k)}(\mathbf{x},\xi)]_{,j}\Big]\,d\Omega_{\mathbf{x}} \\ &= \int\limits_{\Gamma} u_i(\mathbf{x})\sigma_{ij}^{(k)}(\mathbf{x},\xi)n_j(\mathbf{x})d\Gamma_{\mathbf{x}} - \int\limits_{\Omega} u_i(\mathbf{x})[\sigma_{ij}^{(k)}(\mathbf{x},\xi)]_{,j}\,d\Omega_{\mathbf{x}} \\ &= \int\limits_{\Gamma} u_i(\mathbf{x})T_i^{(k)}(\mathbf{x},\xi)d\Gamma_{\mathbf{x}} + \int\limits_{\Omega} u_i(\mathbf{x})\delta(\mathbf{x}-\xi)e_i^{(k)}\,d\Omega_{\mathbf{x}} \quad (3.182)\end{aligned}$$

Taking (3.181) into account when substituting (3.182) in (3.180) and in the equivalent weighted residual form (3.176), respectively, yields with the filtering effect of the δ-function the integral equation for the displacements at arbitrary interior points $\xi \in \Omega$:

$$u_k(\xi) = \int\limits_{\Gamma} \Big[T_i(\mathbf{x})u_i^{(k)}(\mathbf{x},\xi) - u_i(\mathbf{x})T_i^{(k)}(\mathbf{x},\xi)\Big]\,d\Gamma_{\mathbf{x}} + \int\limits_{\Omega} b_i(\mathbf{x})\,u_i^{(k)}(\mathbf{x},\xi)d\Omega_{\mathbf{x}} \quad (3.183)$$

The boundary traction vector $T_i^{(k)}(\mathbf{x},\xi)$ of the fundamental solution may be determined by differentiating the fundamental solution (3.177) and (3.178), respectively, via the definition of the strain tensor and the constitutive relations as:

$$\begin{aligned}T_i^{(k)} &= \frac{1}{2\pi}\frac{1}{2\mu+\lambda}\frac{n_j}{r}\left(\mu(r_{,k}\delta_{ij} - r_{,i}\delta_{jk} - r_{,j}\delta_{ik}) - 2(\mu+\lambda)r_{,i}r_{,j}r_{,k}\right) \quad \text{in } R^2 \quad (3.184) \\ &= \frac{1}{4\pi}\frac{1}{2\mu+\lambda}\frac{n_j}{r^2}\left(\mu(r_{,k}\delta_{ij} - r_{,i}\delta_{jk} - r_{,j}\delta_{ik}) - 3(\mu+\lambda)r_{,i}r_{,j}r_{,k}\right) \quad \text{in } R^3 \quad (3.185)\end{aligned}$$

3.2.1.4 Exercise 12: Strain tensor of the elastostatic fundamental solution

Determine the strain tensor $\varepsilon_{ij}^{(k)}(\mathbf{x},\xi)$ of the fundamental solution $u_i^{(k)}(\mathbf{x},\xi)$ (3.178) in R^3.

Since the displacement integral equations (3.183) contain unknown boundary reactions (at each boundary point two and three in R^2 and R^3, respectively), the arbitrary source point ξ is placed on the boundary Γ to obtain equations which connects, besides the known body force domain integral, only boundary terms with each other. Then, the integral kernels $u_i^{(k)}(\mathbf{x},\xi)$ and $T_i^{(k)}(\mathbf{x},\xi)$ become weakly and stronly singular for $\mathbf{x} = \xi$, respectively, and, similarly, but not identically (see [1], page 71) to the above considered scalar problems, the weakly singular integrals exists as improper integrals, while the those with the strongly singular kernel $T_i^{(k)}(\mathbf{x},\xi)$ produces two contributions: the first are their Cauchy principal values on $\Gamma - \Gamma_\varepsilon$ (the remaining part of the boundary contour (surface) Γ from which the part Γ_ε was cut out by a circle (sphere) of some radius ε, centered at ξ) and the second is an integral over the singular traction $T_i^{(k)}(\mathbf{x},\xi)$ itself on Γ_ε^*, i.e., on the outer (outside the domain Ω) part of the contour (surface) of that circle (sphere) with radius ε which was used for the Cauchy principal value cut-out:

$$\int\limits_{\Gamma-\Gamma_\varepsilon+\Gamma_\varepsilon^*} u_i(\mathbf{x})T_i^{(k)}(\mathbf{x},\xi)d\Gamma_{\mathbf{x}} = \int\limits_{\Gamma-\Gamma_\varepsilon} u_i(\mathbf{x})T_i^{(k)}(\mathbf{x},\xi)d\Gamma_{\mathbf{x}} + u_i(\xi)\int\limits_{\Gamma_\varepsilon^*} T_i^{(k)}(\mathbf{x},\xi)d\Gamma_{\mathbf{x}} \quad (3.186)$$

Since on Γ_ε^* the components n_i of the outward normal unit vector and those of the derivative of r are identic, i.e., $n_i = r_{,i}$ and, therefore, $\partial r/\partial n = 1$, one obtains in R^2 where $(r_{,1}, r_{,2}) = (\cos\varphi, \sin\varphi)$ and $d\Gamma_{\mathbf{x}} = r d\varphi$

$$\begin{aligned} c_{ki}(\xi) &= \int_{\Gamma_\varepsilon^*} T_i^{(k)}(\mathbf{x},\xi) d\Gamma_{\mathbf{x}} = \frac{-1}{2\pi}\frac{1}{2\mu+\lambda}\int_{\Gamma_\varepsilon^*}\frac{1}{r}\left[\mu\delta_{ik} + 2(\mu+\lambda) r_{,i} r_{,k}\right] d\Gamma_{\mathbf{x}} \\ &= \frac{-1}{2\pi}\frac{1}{2\mu+\lambda}\int_{\varphi_1}^{\varphi_2}\begin{bmatrix} \mu + 2(\mu+\lambda)\cos^2\varphi & 2(\mu+\lambda)\sin\varphi\cos\varphi \\ 2(\mu+\lambda)\sin\varphi\cos\varphi & \mu + 2(\mu+\lambda)\sin^2\varphi \end{bmatrix} d\varphi \\ &= \frac{-1}{2\pi}\left\{ \begin{array}{c} (\varphi_2-\varphi_1)\delta_{ki} \\ +\frac{\mu+\lambda}{2\mu+\lambda}\frac{1}{2}\begin{bmatrix} \sin 2\varphi_2 - \sin 2\varphi_1 & -\cos 2\varphi_2 + \cos 2\varphi_1 \\ -\cos 2\varphi_2 + \cos 2\varphi_1 & -\sin 2\varphi_2 + \sin 2\varphi_1 \end{bmatrix} \end{array} \right\} \end{aligned} \tag{3.187}$$

while in R^3 with $(r_{,1}, r_{,2}, r_{,3}) = (\cos\varphi\sin\vartheta, \sin\varphi\sin\vartheta, \cos\vartheta)$ and $d\Gamma_{\mathbf{x}} = r^2\sin\vartheta d\vartheta d\varphi$

$$\begin{aligned} c_{ki}(\xi) &= \int_{\Gamma_\varepsilon^*} T_i^{(k)}(\mathbf{x},\xi) d\Gamma_{\mathbf{x}} = \frac{-1}{2\pi}\frac{1}{2\mu+\lambda}\int_{\Gamma_\varepsilon^*}\frac{1}{r^2}\left[\mu\delta_{ik} + 3(\mu+\lambda) r_{,i} r_{,k}\right] d\Gamma_{\mathbf{x}} \\ &= \frac{-1}{2\pi}\frac{1}{2\mu+\lambda}\int_{\varphi_1}^{\varphi_2}\int_{\vartheta_1}^{\vartheta_2}\left[\mu\delta_{ik} + 3(\mu+\lambda) r_{,i} r_{,k}\right]\sin\vartheta d\vartheta d\varphi \end{aligned} \tag{3.188}$$

In general, the actual values of these factors dependent on the shape of the actually considered boundary. But, when the position of ξ is on a smooth boundary region, i.e., $\varphi_2 - \varphi_1 = \pi$ in R^2 and $\varphi_2 - \varphi_1 = 2\pi$ with $\vartheta_2 - \vartheta_1 = \frac{\pi}{2}$ in R^3, both above integrals (3.187) and (3.188) become simply

$$c_{ki}(\xi) = \frac{-1}{2}\delta_{ki} \tag{3.189}$$

Finally, the boundary integral equations for determining unknown boundary reactions read as ($k = 1, 2$ in R^2 and $k = 1, 2, 3$ in R^3)

$$[\delta_{ki} + c_{ki}(\xi)] u_i(\xi) = \int_{\Gamma-\Gamma_\varepsilon}\left[T_i(\mathbf{x}) u_i^{(k)}(\mathbf{x},\xi) - u_i(\mathbf{x}) T_i^{(k)}(\mathbf{x},\xi)\right] d\Gamma_{\mathbf{x}} + \int_{\Omega'} b_i(\mathbf{x})\, u_i^{(k)}(\mathbf{x},\xi) d\Omega_{\mathbf{x}} \tag{3.190}$$

3.2.2 Indirect integral formulations: the singularity method

As already explain for one-dimensional problems, the basic idea of this indirect method is to introduce **intermediary unknowns - unknown intensities of certain singularity layers. For these singularities, so-called influence functions, i.e., the complete response to the action of a singularity (e.g., a unit point force) must be known everywhere in the considered material. Then, these singularity layers are distributed on a 'fictitious' boundary Γ^+ in a certain small distance d_ε from the real boundary Γ outside the domain Ω and their intensities have to be determined such that the integrated response is equal to the prescribed boundary values on the real boundary Γ.**

In a second step, when these intensities of the singularities are determined, the unknown physical boundary reaction as well as corresponding interior states can easily be found by the same integral equation used in the first step to find the singularity intensities.

3.2.2.1 Sound pressure and sound flux in stationary acoustics (Helmholtz equation)

As known, the stationary sound radiation problem is described by the Helmholtz equation (see (3.159))

$$\Delta p(\mathbf{x}) + \kappa^2 p(\mathbf{x}) = -b(\mathbf{x}) = +\mathrm{i}c\kappa\rho_0 a(\mathbf{x}) \tag{3.191}$$

for the sound pressure distribution $p(\mathbf{x})$ and the fundamental solution solving the equation

$$\Delta p^*(\mathbf{x},\xi) + \kappa^2 p^*(\mathbf{x},\xi) = -\delta(\mathbf{x},\xi)$$

is in R^2 (siehe (3.166))

$$p^*(\mathbf{x},\xi) = \frac{1}{2\pi} K_0(\mathrm{i}\kappa r) \tag{3.192}$$

Hence, the influence function decribing the sound pressure at a point x due to a unit point sound source $a(\mathbf{x}) = \delta(\mathbf{x},\xi)$ at point ξ is found by comparison to be

$$(pa)(\mathbf{x},\xi) = -\mathrm{i}c\kappa\rho_0 \frac{1}{2\pi} K_0(\mathrm{i}\kappa r) \tag{3.193}$$

Since the sound flux $q_n(\mathbf{x})$ in the direction of the normal vector $n_k(\mathbf{x})$ is defined as the normal derivative of the sound pressure, the corresponding influence function for the sound flux is determined by ($r = |\mathbf{x} - \xi|$)

$$(q_n a)(\mathbf{x},\xi) = \frac{\partial\,(pa)(\mathbf{x},\xi)}{\partial x_k} n_k(\mathbf{x}) = -c\kappa^2\rho_0 \frac{1}{2\pi} K_1(\mathrm{i}\kappa r)\frac{\partial r}{\partial x_k} n_k(\mathbf{x}) \tag{3.194}$$

Following the above described idea of the indirect integral equation method, at points ξ on a fictitious boundary Γ^+, which is either enclosing with a certain distance d_ε the real boundary Γ or is coincident to Γ, layers of point sources are introduced with such an intensity $a^*(\xi)$ that the prescribed boundary conditions on the real boundary Γ are satisfied.

For a sound radiation problem where on one part of the boundary Γ_1 the sound pressure $p(\mathbf{x}) = \bar{p}(\mathbf{x})$ and on the remaining part Γ_2 the sound flux $q_n(\mathbf{x}) = \bar{q}_n(\mathbf{x})$ is prescribed, and, moreover, a sound source density $\bar{a}(\mathbf{x})$ is acting in the interior of the considered domain Ω, the two indirect integral equations for determining the unknown point source layer intensity $a^*(\xi)$ are simply given by

$$\begin{aligned} \int_{\Gamma^+} (pa)(\mathbf{x},\xi)a^*(\xi)d\Gamma_\xi + \int_{\Omega} (pa)(\mathbf{x},\xi)\bar{a}(\xi)d\Omega_\xi &= \bar{p}(\mathbf{x}) \text{ for } \mathbf{x} \in \Gamma_1 \\ \int_{\Gamma^+} (q_n a)(\mathbf{x},\xi)a^*(\xi)d\Gamma_\xi + \int_{\Omega} (q_n a)(\mathbf{x},\xi)\bar{a}(\xi)d\Omega_\xi &= \bar{q}_n(\mathbf{x}) \text{ for } \mathbf{x} \in \Gamma_2 \end{aligned}$$

which is explicitly

$$\int\limits_{\Gamma^+} K_0(\mathrm{i}\kappa r)a^*(\xi)d\Gamma_\xi + \int\limits_{\Omega} K_0(\mathrm{i}\kappa r)\bar{a}(\xi)d\Omega_\xi = \frac{2\pi\mathrm{i}\,\bar{p}(\mathbf{x})}{c\kappa\rho_0} \quad \text{for } \mathbf{x} \in \Gamma_1 \tag{3.195}$$

$$n_k(\mathbf{x})\left[\int\limits_{\Gamma^+} \mathrm{i}\kappa K_1(\mathrm{i}\kappa r)\frac{\partial r}{\partial x_k}a^*(\xi)d\Gamma_\xi + \int\limits_{\Omega} \mathrm{i}\kappa K_1(\mathrm{i}\kappa r)\frac{\partial r}{\partial x_k}\bar{a}(\xi)d\Omega_\xi\right] = -\frac{2\pi\mathrm{i}\,\bar{q}_n(\mathbf{x})}{c\kappa\rho_0} \quad \text{for } \mathbf{x} \in \Gamma_2 \tag{3.196}$$

Since the size of the distance d_ε of the fictitious boundary Γ^+ from the real boundary Γ has a large effect on the solution if these integral equations have to be solved numerically, it is mostly better to transfer the fictitious boundary into the real boundary, i.e., $d_\varepsilon \to 0$. In this case, the influence function $(pb)\,(\mathbf{x},\xi)$ becomes with $\mathrm{K}_0(\mathrm{i}\kappa r) \to -\ln(\mathrm{i}\kappa r)$ weakly singular and $(q_n b)\,(\mathbf{x},\xi)$ with $\mathrm{K}_1(\mathrm{i}\kappa r) \to 1/(\mathrm{i}\kappa r)$ strongly singular for $\xi \to \mathbf{x}$.

Hence, similarly to direct integral equation method, the singular point $\mathbf{x} = \xi$ has to be avoided in the integration on $\Gamma^+ \to \Gamma$, but different to the handling there, the integral on Γ^+ is only to split into one on $\Gamma^+ - \Gamma_\varepsilon^+$ and one on Γ_ε^+ (while the integration on the ε-circular arc Γ_ε^{+*} is dropped here), and, for both contributions, the limit towards $\Gamma - \Gamma_\varepsilon$ and Γ_ε is performed.

In the case of the weakly singular integral equation (3.195) for the sound pressure, the integral on $\Gamma_\varepsilon^+ \to \Gamma_\varepsilon$ gives no contribution such that the equation remains formally unchanged:

$$\int\limits_{\Gamma-\Gamma_\varepsilon} K_0(\mathrm{i}\kappa r)a^*(\xi)d\Gamma_\xi + \int\limits_{\Omega} K_0(\mathrm{i}\kappa r)\bar{a}(\xi)d\Omega_\xi = \frac{2\pi\mathrm{i}\,\bar{p}(\mathbf{x})}{c\kappa\rho_0} \quad \text{for } \mathbf{x} \in \Gamma_1 \tag{3.197}$$

The integral with the strongly singular kernel in the equation (3.196) for the sound flux

$$\begin{aligned} n_k(\mathbf{x})\int\limits_{\Gamma^+} \mathrm{i}\kappa K_1(\mathrm{i}\kappa r)\frac{\partial r}{\partial x_k}a^*(\xi)d\Gamma_\xi &= n_k(\mathbf{x})\int\limits_{\Gamma^+-\Gamma_\varepsilon^+} \mathrm{i}\kappa K_1(\mathrm{i}\kappa r)\frac{\partial r}{\partial x_k}a^*(\xi)d\Gamma_\xi \\ &+ n_k(\mathbf{x})a^*(\mathbf{x})\int\limits_{\Gamma_\varepsilon^+} \mathrm{i}\kappa K_1(\mathrm{i}\kappa r)\frac{\partial r}{\partial x_k}d\Gamma_\xi \\ &+ n_k(\mathbf{x})\int\limits_{\Gamma_\varepsilon^+} [a^*(\xi) - a^*(\mathbf{x})]\mathrm{i}\kappa K_1(\mathrm{i}\kappa r)\frac{\partial r}{\partial x_k}d\Gamma_\xi \end{aligned} \tag{3.198}$$

is split in three parts where that on $\Gamma^+ - \Gamma_\varepsilon^+ \to \Gamma - \Gamma_\varepsilon$ exists as Cauchy principal value, that on Γ_ε^+ can be analytically integrated while the third one on $\Gamma_\varepsilon^+ \to \Gamma_\varepsilon$ gives zero assuming at $\mathbf{x} = \xi$ a continous intensity $a^*(\xi)$.

Since here, different to the direct integral equations, the normal vector $n_k(\mathbf{x}) \triangleq (\cos\varphi, \sin\varphi)$ is not that of the integration point ξ, and is only on a smooth boundary uniquely defined, for the evaluation on Γ_ε^+ , a smooth boundary Γ^+ is assumed which is in a distance d_ε parallel to the real boundary Γ. Denoting the distance of the projection of $\mathbf{x}$ on Γ^+ from ξ with s, one can express the distance r between $\mathbf{x} \in \Gamma$ and $\xi \in \Gamma^+$ by $r = \sqrt{s^2 + d_\varepsilon^2}$ and $r_{,i} = r_{,n}n_i + r_{,s}t_i$ can be described by (φ is the angle between the normal direction and the x_1-axis)

$$\begin{bmatrix} r_{,1} \\ r_{,2} \end{bmatrix} = \frac{1}{r}\begin{bmatrix} x_1-\xi_1 \\ x_2-\xi_2 \end{bmatrix} = \frac{1}{r}\begin{bmatrix} -d_\varepsilon\cos\varphi - s\sin\varphi \\ -d_\varepsilon\sin\varphi + s\cos\varphi \end{bmatrix} = -\frac{d_\varepsilon}{r}\begin{bmatrix} n_1(\mathbf{x}) \\ n_2(\mathbf{x}) \end{bmatrix} + \frac{s}{r}\begin{bmatrix} t_1(\mathbf{x}) \\ t_2(\mathbf{x}) \end{bmatrix} \tag{3.199}$$

Applying this and the approximation of $K_1(\mathrm{i}\kappa r) \approx 1/(\mathrm{i}\kappa r)$ for small distances r, gives, since $n_k(\mathbf{x})n_k(\mathbf{x}) = 1$ and $n_k(\mathbf{x})t_k(\mathbf{x}) = 0$

$$\begin{aligned} n_k(\mathbf{x})a^*(\mathbf{x}) \lim_{d_\varepsilon\to 0} \int_{\Gamma_\varepsilon^+} \mathrm{i}\kappa K_1(\mathrm{i}\kappa r)\frac{\partial r}{\partial x_k}d\Gamma_\xi &= a^*(\mathbf{x})n_k(\mathbf{x}) \lim_{d_\varepsilon\to 0} \int_{-\varepsilon}^{\varepsilon} \left\{ \mathrm{i}\kappa\frac{1}{\mathrm{i}\kappa r}\left\{ \begin{array}{c} -\frac{d_\varepsilon}{r}n_k(\mathbf{x}) \\ +\frac{s}{r}t_k(\mathbf{x}) \end{array} \right\}\right\} ds \\ &= -a^*(\mathbf{x}) \lim_{d_\varepsilon\to 0} \left\{ \int_{-\varepsilon}^{\varepsilon} \frac{d_\varepsilon}{s^2+d_\varepsilon^2} ds \right\} \\ &= -a^*(\mathbf{x}) \lim_{d_\varepsilon\to 0} \left[\arctan\frac{s}{d_\varepsilon} \right]_{-\varepsilon}^{\varepsilon} \\ &= -a^*(\mathbf{x}) \lim_{d_\varepsilon\to 0} \left[\arctan\frac{\varepsilon}{d_\varepsilon} - \arctan\frac{-\varepsilon}{d_\varepsilon} \right] \\ &= -\pi a^*(\mathbf{x}) \end{aligned} \tag{3.200}$$

Finally, the following singular version of (3.196) for satisfying prescribed sound flux conditions is obtained

$$\frac{a^*(\mathbf{x})}{2} - \frac{n_k(\mathbf{x})}{2\pi}\left[\int_{\Gamma-\Gamma_\varepsilon} \mathrm{i}\kappa K_1(\mathrm{i}\kappa r)\frac{\partial r}{\partial x_k}a^*(\xi)d\Gamma_\xi + \int_\Omega \mathrm{i}\kappa K_1(\mathrm{i}\kappa r)\frac{\partial r}{\partial x_k}\bar{a}(\xi)d\Omega_\xi \right] = \frac{\mathrm{i}\bar{q}_n(\mathbf{x})}{c\kappa\rho_0} \text{for } \mathbf{x} \in \Gamma_2 \tag{3.201}$$

When by solving the two equations (3.197) and (3.201) the adequate intensities $a^*(\xi)$ are found to represent the prescribed boundary values, the same two equations can be used to find the unknown boundary reactions $p(\mathbf{x})$ on Γ_2 and $q_n(\mathbf{x})$ on Γ_1 , and the two equations (3.195) and (3.196) (due to $\Gamma^+ \to \Gamma$ by integration on Γ) for finding the sound pressure and the sound flux at any interior point $\mathbf{x}$.

3.2.2.2 Displacements and stresses in elastostatics (Navier equations)

The Navier equations (3.175)

$$\mu\frac{\partial^2 u_i(\mathbf{x})}{\partial x_j\partial x_j} + (\lambda+\mu)\frac{\partial^2 u_j(\mathbf{x})}{\partial x_j\partial x_i} = -b_i(\mathbf{x}) \tag{3.202}$$

describe the displacements of an elastic body under its dead weight $b_i(\mathbf{x})$ and as a result of prescribed displacements $u_i(\mathbf{x}) = \bar{u}_i(\mathbf{x})$ on a part of the boundary Γ_1 and/or of prescribed boundary tractions $T_i(\mathbf{x}) = \bar{T}_i(\mathbf{x})$ on the remaining boundary part Γ_2.

Besides these, other dynamic or geometric states, e.g., point forces, single moments, or dislocations, may occur and produce displacements and deformations.

The solutions $u_i^{(k)}(\mathbf{x},\xi)$ of the equations

$$\mu\frac{\partial^2 u_i^{(k)}(\mathbf{x},\xi)}{\partial x_j\partial x_j} + (\lambda+\mu)\frac{\partial^2 u_j^{(k)}(\mathbf{x},\xi)}{\partial x_j\partial x_i} = -b_i^*(\mathbf{x},\xi) = -\delta(\mathbf{x}-\xi)e_i^{(k)} \tag{3.203}$$

i.e., the displacements (see (3.177))

$$u_i^{(k)}(\mathbf{x},\xi) = \frac{1}{4\pi}\frac{1}{2\mu+\lambda}\left[-(3+\frac{\lambda}{\mu})\delta_{ik}\ln r + (1+\frac{\lambda}{\mu})r_{,i}r_{,k}\right] \text{ in } R^2 \tag{3.204}$$

are the reactions at a point $\mathbf{x}$ on a unit point force $b_i^*(\mathbf{x},\xi) = F_i(\mathbf{x},\xi) = \delta(\mathbf{x}-\xi)e_i^{(k)}$ applied at a given fixed point $\xi \in \Omega$ along the k-direction. Hence, this fundamental solution is also a influence function representing displacements due to unit point forces

$$(uF)_{i.k}(\mathbf{x},\xi) = u_i^{(k)}(\mathbf{x},\xi) \tag{3.205}$$

Then, following the idea of the indirect integral equation method, point force layers with unknown intensity $F_k^*(\xi)$ along the k-direction at points ξ on a fictitious boundary Γ^+ enclosing with a distance d_ε the real boundary Γ are introduced which produce together with the known prescribed dead weight loading $\bar{b}_k(\xi)$, $\xi \in\Omega$, displacements at points $\mathbf{x}$

$$u_i(\mathbf{x}) = \int\limits_{\Gamma^+} (uF)_{i.k}(\mathbf{x},\xi)F_k^*(\xi)d\Gamma_\xi + \int\limits_{\Omega} (uF)_{i.k}(\mathbf{x},\xi)\bar{b}_k(\xi)d\Omega_\xi \tag{3.206}$$

For representing prescribed displacements $\bar{u}_i(\mathbf{x})$ on a part Γ_1 of the real boundary Γ, the intensities $F_k^*(\xi)$ have to satisfy the indirect boundary integral equation:

$$\int\limits_{\Gamma^+} (uF)_{i.k}(\mathbf{x},\xi)F_k^*(\xi)d\Gamma_\xi + \int\limits_{\Omega} (uF)_{i.k}(\mathbf{x},\xi)\bar{b}_k(\xi)d\Omega_\xi = \bar{u}_i(\mathbf{x}) \text{ for } \mathbf{x} \in\Gamma_1 \tag{3.207}$$

When on the remaining part $\Gamma_2 = \Gamma - \Gamma_1$ boundary tractions $\bar{T}_i(\mathbf{x})$ are prescribed, influence functions representing boundary tractions are needed which can be determined by differentiating the displacement influence function (3.205), respectively (3.204), via the definition of the strain tensor and the constitutive relations (see (3.184))

$$\begin{aligned}(TF)_{i.k}(\mathbf{x},\xi) &= \sigma_{ij}^{(k)}(\mathbf{x},\xi)n_j(\mathbf{x}) = \left(2\mu\varepsilon_{ij}^{(k)}(\mathbf{x},\xi) + \lambda\delta_{ij}\varepsilon_{ll}^{(k)}(\mathbf{x},\xi)\right)n_j(\mathbf{x}) \\ &= \frac{1}{2\pi}\frac{\mu}{2\mu+\lambda}\frac{1}{r}\left[\begin{array}{c} r_{,k}\delta_{ij} - r_{,i}\delta_{jk} - r_{,j}\delta_{ik} \\ -2(1+\frac{\lambda}{\mu})r_{,i}r_{,j}r_{,k}\end{array}\right]n_j(\mathbf{x})\end{aligned} \tag{3.208}$$

This gives the indirect boundary integral equations for representing boundary tractions as

$$\int\limits_{\Gamma^+} (TF)_{i.k}(\mathbf{x},\xi)F_k^*(\xi)d\Gamma_\xi + \int\limits_{\Omega} (TF)_{i.k}(\mathbf{x},\xi)\bar{b}_k(\xi)d\Omega_\xi = \bar{T}_i(\mathbf{x}) \text{ for } \mathbf{x} \in\Gamma_2 \tag{3.209}$$

In this regular version of the indirect boundary equations (3.207) und (3.209) for a mixed elastostatic boundary value problem, the distance d_ε between the fictitious boundary Γ^+ and the real boundary Γ has a large effect on a numerically determined solution.

Therefore, it is advantageous to shift with $d_\varepsilon \to 0$ the fictitious boundary Γ^+ towards the real boundary Γ. Then, the integral kernels $(uF)_{i.k}(\mathbf{x},\xi)$ and $(TF)_{i.k}(\mathbf{x},\xi)$ become for $\xi \to \mathbf{x}$ weakly singular and strongly singular, as $\ln r$ and $1/r$, respectively.

Hence, similarly to direct integral equation method, the singular point $\mathbf{x} = \xi$ has to be avoided in the integration on $\Gamma^+ \to \Gamma$, but different to the handling there, the integral on Γ^+ is only to split into one on $\Gamma^+ - \Gamma_\varepsilon^+$ and one on Γ_ε^+ (while the integration on a ε-circular arc Γ_ε^{+*}, equivalent to Γ_ε^* in the diect method, is dropped here), and, for both contributions, the limit towards $\Gamma - \Gamma_\varepsilon$ and Γ_ε is performed.

In the case of the weakly singular integral equation (3.207) for the displacements, the integral on $\Gamma_\varepsilon^+ \to \Gamma_\varepsilon$ gives no contribution such that the equation remains formally unchanged:

$$\int_{\Gamma-\Gamma_\varepsilon} (uF)_{i.k}(\mathbf{x},\xi) F_k^*(\xi) d\Gamma_\xi + \int_\Omega (uF)_{i.k}(\mathbf{x},\xi) \bar{b}_k(\xi) d\Omega_\xi = \bar{u}_i(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma_1 \tag{3.210}$$

The integral with the strongly singular kernel in the equation (3.209) for the boundary tractions

$$\int_{\Gamma^+} (TF)_{i.k}(\mathbf{x},\xi) F_k^*(\xi) d\Gamma_\xi = \frac{n_j(\mathbf{x})}{2\pi} \frac{\mu}{2\mu+\lambda} \int_{\Gamma^+} \frac{1}{r} \begin{bmatrix} r_{,k}\delta_{ij} - r_{,i}\delta_{jk} - r_{,j}\delta_{ik} \\ -2(1+\frac{\lambda}{\mu}) r_{,i} r_{,j} r_{,k} \end{bmatrix} F_k^*(\xi) d\Gamma_\xi \tag{3.211}$$

$$\begin{aligned} &= \int_{\Gamma^+-\Gamma_\varepsilon^+} (TF)_{i.k}(\mathbf{x},\xi) F_k^*(\xi) d\Gamma_\xi \\ &\quad + F_k^*(\mathbf{x}) \int_{\Gamma_\varepsilon^+} (TF)_{i.k}(\mathbf{x},\xi) d\Gamma_\xi \\ &\quad + \int_{\Gamma_\varepsilon^+} [F_k^*(\xi) - F_k^*(\mathbf{x})] (TF)_{i.k}(\mathbf{x},\xi) d\Gamma_\xi \end{aligned} \tag{3.212}$$

is split in three parts where that on $\Gamma^+ - \Gamma_\varepsilon^+ \to \Gamma - \Gamma_\varepsilon$ exists as Cauchy principal value, that on Γ_ε^+ can be analytically integrated while the third one on $\Gamma_\varepsilon^+ \to \Gamma_\varepsilon$ gives zero assuming at $\mathbf{x} = \xi$ a continous intensity $F_k^*(\xi)$.

Since here, different to the direct integral equations, the normal vector $n_k(\mathbf{x}) \triangleq (\cos\varphi, \sin\varphi)$ is not that of the integration point ξ, and is only on a smooth boundary uniquely defined, for the evaluation on Γ_ε^+ , a smooth boundary Γ^+ is assumed which is in a distance d_ε parallel to the real boundary Γ. Denoting the distance of the projection of $\mathbf{x}$ on Γ^+ from ξ with s, one can express the distance r between $\mathbf{x} \in \Gamma$ and $\xi \in \Gamma^+$ by $r = \sqrt{s^2 + d_\varepsilon^2}$ and $r_{,i} = r_{,n} n_i + r_{,s} t_i$ can be described by (see (3.199))

$$r_{,i} = -\frac{d_\varepsilon}{r} n_i(\mathbf{x}) - \frac{s}{r} t_i(\mathbf{x}) \tag{3.213}$$

Following the evaluation in (3.200), one obtains for the essential singular part

$$\lim_{d_\varepsilon \to 0} \int_{\Gamma_\varepsilon^+} \frac{1}{r} \frac{\partial r}{\partial x_k} d\Gamma_\xi = \lim_{d_\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \left\{ \frac{-1}{r} \left[\frac{d_\varepsilon}{r} n_k(\mathbf{x}) + \frac{s}{r} t_k(\mathbf{x}) \right] \right\} ds = -\pi n_k(\mathbf{x}) \tag{3.214}$$

and with $\lim_{d_\varepsilon \to 0}$ for

$$
\begin{aligned}
\int_{\Gamma_\varepsilon^+} \frac{1}{r}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k} d\Gamma_\xi &= \int_{-\varepsilon}^{\varepsilon} \left\{ \frac{-1}{r} \begin{bmatrix} \frac{d_\varepsilon}{r} n_i(\mathbf{x}) \\ +\frac{s}{r} t_i(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \frac{d_\varepsilon}{r} n_j(\mathbf{x}) \\ +\frac{s}{r} t_j(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \frac{d_\varepsilon}{r} n_k(\mathbf{x}) \\ +\frac{s}{r} t_k(\mathbf{x}) \end{bmatrix} \right\} ds \\
&= \int_{-\varepsilon}^{\varepsilon} \left\{ \frac{-1}{r^4} \begin{bmatrix} d_\varepsilon^3 n_i n_j n_k + d_\varepsilon^2 s \left(n_i n_j t_k + n_i t_j n_k + t_i n_j n_k \right) \\ + d_\varepsilon s^2 \left(n_i t_j t_k + t_i n_j t_k + t_i t_j n_k \right) + s^3 t_i t_j t_k \end{bmatrix} \right\} ds \\
&= - \left\{ \begin{array}{c} \left[\frac{s d_\varepsilon}{2(d_\varepsilon^2+\delta^2)} + \frac{1}{2} \arctan \frac{s}{d_\varepsilon} \right]_{-\varepsilon}^{\varepsilon} n_i n_j n_k \\ + \left[\frac{-d_\varepsilon^2}{s^2+d_\varepsilon^2} \right]_{-\varepsilon}^{\varepsilon} \left(n_i n_j t_k + n_i t_j n_k + t_i n_j n_k \right) \\ + \left[\frac{-s d_\varepsilon}{2(s^2+d_\varepsilon^2)} + \frac{1}{2} \arctan \frac{s}{d_\varepsilon} \right]_{-\varepsilon}^{\varepsilon} \left(n_i t_j t_k + t_i n_j t_k + t_i t_j n_k \right) \\ + \left[\frac{d_\varepsilon^2}{s^2+d_\varepsilon^2} + \frac{1}{2} \ln(s^2 + d_\varepsilon^2) \right]_{-\varepsilon}^{\varepsilon} t_i t_j t_k \end{array} \right\} \\
&= -\frac{\pi}{2} \left\{ n_i n_j n_k + n_i t_j t_k + t_i n_j t_k + t_i t_j n_k \right\} \quad \text{for } d_\varepsilon \to 0
\end{aligned}
\tag{3.215}
$$

This yields with $n_j n_j = 1$, $t_j n_j = 0$, and $n_i n_k + t_i t_k = \delta_{ik}$ for

$$
\begin{aligned}
\lim_{d_\varepsilon \to 0} \int_{\Gamma_\varepsilon^+} (TF)_{i.k}(\mathbf{x},\xi) d\Gamma_\xi &= \frac{1}{2\pi}\frac{\mu}{2\mu+\lambda} \left\{ \begin{array}{c} -\pi(n_k n_i - n_i n_k - n_j \delta_{ik} n_j) \\ +2(1+\frac{\lambda}{\mu})\frac{\pi}{2}(n_i n_j n_k + n_i t_j t_k + t_i n_j t_k + t_i t_j n_k) n_j \end{array} \right\} \\
&= \frac{1}{2}\frac{\mu}{2\mu+\lambda} \left\{ \delta_{ik} + (1+\frac{\lambda}{\mu})(n_i n_k + t_i t_k) \right\} = \frac{1}{2}\delta_{ik}
\end{aligned}
\tag{3.216}
$$

Finally, the singular version of the indirect boundary integral equation for the boundary tractions is obtained to be

$$
\frac{1}{2}F_i^*(\mathbf{x}) + \int_{\Gamma-\Gamma_\varepsilon} (TF)_{i.k}(\mathbf{x},\xi) F_k^*(\xi) d\Gamma_\xi + \int_{\Omega} (TF)_{i.k}(\mathbf{x},\xi) \bar{b}_k(\xi) d\Omega_\xi = \bar{T}_i(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma_2
\tag{3.217}
$$

Are the singularity layer intensities $F_k^*(\xi)$ determined by solving the integral equations (3.210) and (3.217), in a second step, the unknown boundary reactions, i.e., $u_i(\mathbf{x})$ for $\mathbf{x} \in \Gamma_2$ and $T_i(\mathbf{x})$ for $\mathbf{x} \in \Gamma_1$ can easily found be evaluating the integral equations (3.210) and (3.217), respectively.

Then, it is also possible to analyse the stresses at interior points by evaluating with the determined intensities $F_k^*(\xi)$ the integral relation

$$
\sigma_{ij}(\mathbf{x}) = \int_{\Gamma} (\sigma F)_{ij.k}(\mathbf{x},\xi) F_k^*(\xi) d\Gamma_\xi + \int_{\Omega} (\sigma F)_{ij.k}(\mathbf{x},\xi) \bar{b}_k(\xi) d\Omega_\xi \quad \text{for } \mathbf{x} \in \Omega
\tag{3.218}
$$

where the influence function $(\sigma F)_{ij.k}(\mathbf{x},\xi)$ for the stresses is easily found from that of the boundary tractions (3.208) to be

$$
(\sigma F)_{ij.k}(\mathbf{x},\xi) = \frac{1}{2\pi}\frac{\mu}{2\mu+\lambda}\frac{1}{r} \left(r_{,k}\delta_{ij} - r_{,i}\delta_{jk} - r_{,j}\delta_{ik} - 2(1+\frac{\lambda}{\mu}) r_{,i} r_{,j} r_{,k} \right).
\tag{3.219}
$$

3.2.3 Integral formulation with Green's functions

As defined above, the so-called *Green's function* $G^*(\mathbf{x}, \xi)$ of a boundary value problem is a special fundamental solution, i.e.,

$$L(D)G^*(\mathbf{x}, \xi) = \delta(\mathbf{x}, \xi)$$

which satisfies homogeneous conditions for those boundary states which are prescribed in the actual problem, i.e.,

$$\begin{aligned} E(G^*(\mathbf{x}, \xi)) &= 0 \text{ for } \mathbf{x} \in \Gamma_1 \\ N(G^*(\mathbf{x}, \xi)) &= 0 \text{ for } \mathbf{x} \in \Gamma_2 \end{aligned}$$

The meaning of this definition shall now be demonstrated for some explicit problems.

3.2.3.1 Temperature distribution in stationary heat conduction

As described above in equation (3.158), the direct form of the integral equation for the temperature $\Theta(\xi)$ on the boundary, i.e., for $\xi \in \Gamma$, reads

$$\frac{\Delta\Omega(\xi)}{2\pi}\Theta(\xi) = -\int\limits_{\Gamma-\Gamma_\varepsilon} \frac{1}{\lambda_0}\left[q_n(\mathbf{x})\Theta^*(\mathbf{x}, \xi) - \Theta(\mathbf{x})q_n^*(\mathbf{x}, \xi)\right] d\Gamma_{\mathbf{x}} - \int\limits_{\Omega} \frac{W_q(\mathbf{x})}{\lambda_0}\Theta^*(\mathbf{x}, \xi)d\Omega_{\mathbf{x}} \tag{3.220}$$

while it is given in the interior, i.e., at points $\xi \in \Omega$ (see equation (3.151)) by

$$\Theta(\xi) = -\int\limits_{\Gamma} \frac{1}{\lambda_0}\left(q_n(\mathbf{x})\Theta^*(\mathbf{x}, \xi) - \Theta(\mathbf{x})q_n^*(\mathbf{x}, \xi)\right) d\Gamma_{\mathbf{x}} - \int\limits_{\Omega} \frac{W_q(\mathbf{x})}{\lambda_0}\Theta^*(\mathbf{x}, \xi)d\Omega_{\mathbf{x}} \tag{3.221}$$

When the actual boundary value problem is defined by a prescribed temperature $\Theta(\mathbf{x}) = \bar{\Theta}(\mathbf{x})$ on a part of the boundary Γ_1 and/or of prescribed temperature flux $q_n(\mathbf{x}) = \bar{q}_n(\mathbf{x})$ on the remaining boundary part Γ_2, the more detailed form of (3.226)

$$\begin{aligned} \Theta(\xi) \;=\; &-\int\limits_{\Gamma_1} \frac{1}{\lambda_0}\left[q_n(\mathbf{x})\Theta^*(\mathbf{x}, \xi) - \bar{\Theta}(\mathbf{x})q_n^*(\mathbf{x}, \xi)\right] d\Gamma_{\mathbf{x}} - \int\limits_{\Gamma_2} \frac{1}{\lambda_0}\left[\bar{q}_n(\mathbf{x})\Theta^*(\mathbf{x}, \xi) - \Theta(\mathbf{x})q_n^*(\mathbf{x}, \xi)\right] d\Gamma_{\mathbf{x}} \\ &-\int\limits_{\Omega} \frac{W_q(\mathbf{x})}{\lambda_0}\,\Theta^*(\mathbf{x}, \xi)d\Omega_{\mathbf{x}} \end{aligned} \tag{3.222}$$

shows explicitly the unknown boundary reactions $q_n(\mathbf{x})$ on Γ_1 and $\Theta(\mathbf{x})$ on Γ_2. Hence, it is obvious that the temperature $\Theta(\xi)$ could directly be determined by this integral if the respective integrands containing unknowns would be zero, i.e., if one finds a special fundamental solution $G(\mathbf{x}, \xi)$ which satisfies additionally the conditions

$$G(\mathbf{x}, \xi) = 0 \text{ for } \mathbf{x} \in \Gamma_1 \text{ and } \quad q_n(G(\mathbf{x}, \xi)) = 0 \text{ for } \mathbf{x} \in \Gamma_2 \tag{3.223}$$

Then, the temperature at any interior point $\xi \in \Omega$ is expressed by

$$\Theta(\xi) = \int_{\Gamma_1} \frac{1}{\lambda_0}\bar{\Theta}(\mathbf{x})q_n(G(\mathbf{x},\xi))d\Gamma_{\mathbf{x}} - \int_{\Gamma_2} \frac{1}{\lambda_0}\bar{q}_n(\mathbf{x})G(\mathbf{x},\xi)d\Gamma_{\mathbf{x}} - \int_{\Omega} \frac{W_q(\mathbf{x})}{\lambda_0}G(\mathbf{x},\xi)d\Omega_{\mathbf{x}} \quad (3.224)$$

without determining first the unknown boundary reactions by solving the boundary integral equation (3.225).

3.2.3.2 Sound pressure in stationary acoustics

As described above in equation (3.174), the direct form of the integral equation for the sound pressure $p(\xi)$ on the boundary, i.e., for $\xi \in \Gamma$, reads

$$\frac{\Delta\varphi(\xi)}{2\pi}p(\xi) = \int_{\Gamma-\Gamma_\varepsilon} \left[q_n(\mathbf{x})p^*(\mathbf{x},\xi) - p(\mathbf{x})q_n^*(\mathbf{x},\xi)\right] d\Gamma_{\mathbf{x}} + \int_{\Omega} b(\mathbf{x})\, p^*(\mathbf{x},\xi)d\Omega_{\mathbf{x}} \quad (3.225)$$

while it is given in the interior, i.e., at points $\xi \in \Omega$ (see equation (3.170)) by

$$p(\xi) = \int_{\Gamma} \left[q_n(\mathbf{x})p^*(\mathbf{x},\xi) - p(\mathbf{x})q_n^*(\mathbf{x},\xi)\right] d\Gamma_{\mathbf{x}} + \int_{\Omega} b(\mathbf{x})\, p^*(\mathbf{x},\xi)d\Omega_{\mathbf{x}} \quad (3.226)$$

When the actual boundary value problem is defined by a prescribed pressure $p(\mathbf{x}) = \bar{p}(\mathbf{x})$ on a part of the boundary Γ_1 and/or of prescribed sound flux $q_n(\mathbf{x}) = \bar{q}_n(\mathbf{x})$ on the remaining boundary part Γ_2, the more detailed form of (3.226)

$$\begin{aligned} p(\xi) \;=\; & \int_{\Gamma_1} \left[q_n(\mathbf{x})p^*(\mathbf{x},\xi) - \bar{p}(\mathbf{x})q_n^*(\mathbf{x},\xi)\right] d\Gamma_{\mathbf{x}} + \int_{\Gamma_2} \left[\bar{q}_n(\mathbf{x})p^*(\mathbf{x},\xi) - p(\mathbf{x})q_n^*(\mathbf{x},\xi)\right] d\Gamma_{\mathbf{x}} \\ & + \int_{\Omega} b(\mathbf{x})\, p^*(\mathbf{x},\xi)d\Omega_{\mathbf{x}} \end{aligned} \quad (3.227)$$

shows explicitly the unknown boundary reactions $q_n(\mathbf{x})$ on Γ_1 and $p(\mathbf{x})$ on Γ_2. Hence, it is obvious that the sound pressure $p(\xi)$ could directly be determined by this integral if the respective integrands containing unknowns would be zero, i.e., if one finds a special fundamental solution $G(\mathbf{x},\xi)$ which satisfies additionally the conditions

$$G(\mathbf{x},\xi) = 0 \;\text{ for } \mathbf{x} \in \Gamma_1 \;\text{ and }\; q_n(G(\mathbf{x},\xi)) = 0 \;\text{ for } \mathbf{x} \in \Gamma_2 \quad (3.228)$$

Then, the sound pressure at any interior point $\xi \in \Omega$ is expressed by

$$p(\xi) = -\int_{\Gamma_1} \bar{p}(\mathbf{x})q_n(G(\mathbf{x},\xi))d\Gamma_{\mathbf{x}} + \int_{\Gamma_2} \bar{q}_n(\mathbf{x})G(\mathbf{x},\xi)d\Gamma_{\mathbf{x}} + \int_{\Omega} b(\mathbf{x})\, G(\mathbf{x},\xi)d\Omega_{\mathbf{x}} \quad (3.229)$$

without determining first the unknown boundary reactions by solving the boundary integral equation (3.225).

3.2.3.3 Displacements in elastic bodies

As described above in equation (3.190), the direct form of the integral equation for displacements $u_i(\xi)$ on the boundary, i.e., for $\xi \in \Gamma$, reads

$$[\delta_{ki} + c_{ki}(\xi)]u_i(\xi) = \int\limits_{\Gamma-\Gamma_\varepsilon} \left[T_i(\mathbf{x})u_i^{(k)}(\mathbf{x},\xi) - u_i(\mathbf{x})T_i^{(k)}(\mathbf{x},\xi)\right] d\Gamma_\mathbf{x} + \int\limits_{\Omega'} b_i(\mathbf{x})\, u_i^{(k)}(\mathbf{x},\xi) d\Omega_\mathbf{x} \tag{3.230}$$

while for interior displacements $u_i(\xi)$ with $\xi \in \Omega$ holds (see equation (3.183))

$$u_k(\xi) = \int\limits_{\Gamma} \left[T_i(\mathbf{x})u_i^{(k)}(\mathbf{x},\xi) - u_i(\mathbf{x})T_i^{(k)}(\mathbf{x},\xi)\right] d\Gamma_\mathbf{x} + \int\limits_{\Omega} b_i(\mathbf{x})\, u_i^{(k)}(\mathbf{x},\xi) d\Omega_\mathbf{x} \tag{3.231}$$

When the boundary value problem is defined by prescribed displacements $u_i(\mathbf{x}) = \bar{u}_i(\mathbf{x})$ on a part of the boundary Γ_1 and/or of prescribed boundary tractions $T_i(\mathbf{x}) = \bar{T}_i(\mathbf{x})$ on the remaining boundary part Γ_2, the more detailed form of (3.183)

$$\begin{aligned} u_k(\xi) \;=\; & \int\limits_{\Gamma_1} \left[T_i(\mathbf{x})u_i^{(k)}(\mathbf{x},\xi) - \bar{u}_i(\mathbf{x})T_i^{(k)}(\mathbf{x},\xi)\right] d\Gamma_\mathbf{x} + \int\limits_{\Gamma_2} \left[\bar{T}_i(\mathbf{x})u_i^{(k)}(\mathbf{x},\xi) - u_i(\mathbf{x})T_i^{(k)}(\mathbf{x},\xi)\right] d\Gamma_\mathbf{x} \\ & + \int\limits_{\Omega} b_i(\mathbf{x})\, u_i^{(k)}(\mathbf{x},\xi) d\Omega_\mathbf{x} \end{aligned} \tag{3.232}$$

shows the unknown boundary reactions $T_i(\mathbf{x})$ on Γ_1 and $u_i(\mathbf{x})$ on Γ_2. Hence, it is obvious that the displacements $u_k(\xi)$ could directly be determined by these integrals if the respective integrands containing unknowns would be zero, i.e., if one finds a special fundamental solution $G_i^{(k)}(\mathbf{x},\xi)$ which satisfies additionally the conditions

$$G_i^{(k)}(\mathbf{x},\xi) = 0 \;\text{ for } \mathbf{x} \in \Gamma_1 \;\text{ and }\; T_i(G_j^{(k)}(\mathbf{x},\xi)) = 0 \;\text{ for } \mathbf{x} \in \Gamma_2 \tag{3.233}$$

Then, the displacement at any position $\xi \in \Omega$ is determined by

$$\begin{aligned} u_k(\xi) \;=\; & -\int\limits_{\Gamma_1} \bar{u}_i(\mathbf{x})T_i(G_j^{(k)}(\mathbf{x},\xi))d\Gamma_\mathbf{x} + \int\limits_{\Gamma_2} \bar{T}_i(\mathbf{x})G_i^{(k)}(\mathbf{x},\xi)d\Gamma_\mathbf{x} \\ & + \int\limits_{\Omega} b_i(\mathbf{x})\, G_i^{(k)}(\mathbf{x},\xi) d\Omega_\mathbf{x} \end{aligned} \tag{3.234}$$

4 Numerical solution of boundary integral equations: The boundary element method

In the case of two- and three-dimensional problems, in general, boundary integral equations can only approximatively be solved, i.e., approximations have to be introduced for the boundary and/or the state functions, integrations are performed numerically and not analytically, the integral equations are satisfied only pointwise (i.e., point collocation), and, finally, the resulting system of algebraic equations is solved also only numerically either by a direct solver (Gauss elimination) or iteratively.

These steps are explained in the following sections.

4.1 Approximation of the boundary and of boundary states

The first discretization step is the partition of the boundary curve Γ in non-intersecting so-called *boundary elements* Γ^e $(e = 1, .., m)$ by introducing on the boundary so-called *node points* $\mathbf{x}^{le}$ $(l = 1, .., n)$ where the nodes on the interface between two neighbour elements Γ^e and Γ^{e+1} have the same coordinates but different node indices, e.g., $\mathbf{x}^{ne} = \mathbf{x}^{1e+1}$ in R^2.

Inside these elements, both the boundary curve (if it is not straight) and the state functions of the boundary value problem have to be approximated usually by polynoms as exact as wanted or needed. The polynomial order for approximating the geometry and the boundary states may be different; if these orders are chosen to be equal, one uses a so-called *isoparametric concept*.

Remark: Of course, exactly the same considerations led in past to the concepts and techniques now commonly used in finite element methods. The usual concept of boundary elements is a mere transposition of that of a finite element and, hence, based on the use of nodes and shape functions.

4.1.1 On boundary curves in R^2

For the geometric approximation of the boundary, usually a mapping of each physical boundary element Γ^e onto a *parent* element Δ_e in a parameter space is introduced where the parent element assumes a simple shape, i.e., a line segment $\eta \in [0, 1]$ or $\eta \in [-1, 1]$:

$$\eta \in \Delta_e \to \mathbf{x}^e(\eta) = \sum_{l=1}^{n} \mathbf{x}^{le} N_l^n(\eta) \ \text{ with } \ 0 \leq \eta \leq 1 \tag{4.1}$$

where the index l $(1 \leq l \leq n)$ defines the local numbering of the nodes on element e. The n shape functions, usually of polynomial type and at least linear, since the boundary approximation should be continous, are subjected to the following restrictions:

$$N_p^n(\eta_q) = \delta_{pq} \quad \text{and} \quad \sum_{l=1}^{n} N_l^n(\eta) = 1 \;\; \forall \eta \in \Delta_e \tag{4.2}$$

where $\eta_q \in \Delta_e$ is the *antecedent* of the physical node $\mathbf{x}^{qe}$. Of course, the specific choice of shape functions and the number n of nodes defining the element are related, e.g.,

for a **linear** approximation is $n = 2$ with initial and end node $\mathbf{x}^{1e}$ and $\mathbf{x}^{2e}$, respectively, the shape functions are defined as

$$N_1^2(\eta) = 1 - \eta;\; N_2^2(\eta) = \eta \quad \text{for} \;\; \eta \in \Delta_e = [0, 1] \tag{4.3}$$

and the boundary element is approximated by

$$\mathbf{x}^e(\eta) = \mathbf{x}^{1e}(1 - \eta) + \mathbf{x}^{2e}\eta \tag{4.4}$$

for a **quadratic** approximation is $n = 3$ with initial, middle, and end node x_i^{1e}, x_i^{2e} and x_i^{3e}, respectively, the shape functions are

$$\begin{aligned} N_1^3(\eta) &= (1 - \eta)(1 - 2\eta); \\ N_2^3(\eta) &= 4\eta(1 - \eta); \\ N_3^3(\eta) &= \eta(2\eta - 1) \quad \text{for} \;\; \eta \in \Delta_e = [0, 1] \end{aligned} \tag{4.5}$$

and the boundary element is approximated by

$$\mathbf{x}^e(\eta) = \mathbf{x}^{1e}(1 - \eta)(1 - 2\eta) + \mathbf{x}^{2e}4\eta(1 - \eta) + \mathbf{x}^{3e}\eta(2\eta - 1) \tag{4.6}$$

Remark: It should be mentioned that on a straight boundary element when x_i^{2e} is taken to be the central node, i.e., is taken as the arithmetic mean of the initial and the end node coordinates, $\mathbf{x}^{2e} = 0.5(\mathbf{x}^{1e} + \mathbf{x}^{3e})$, the quadratic approximation (4.6) is reduced to

$$\begin{aligned} \mathbf{x}^e(\eta) &= \mathbf{x}^{1e}(1 - \eta)(1 - 2\eta) + 0.5(\mathbf{x}^{1e} + \mathbf{x}^{3e})4\eta(1 - \eta) + \mathbf{x}^{3e}\eta(2\eta - 1) \\ &= \mathbf{x}^{1e}(1 - \eta) + \mathbf{x}^{3e}\eta \end{aligned}$$

i.e., to the linear approximation (4.4)

4.1.1.1 Exercises 13: Shape functions with the local coordinate $-1 \leq \eta \leq 1$

Determine the linear and the quadratic shape functions (corresponding to (4.3) and (4.5), respectively) when the local coordinate η is defined in the range $-1 \leq \eta \leq 1$.

In the same way, the **boundary states** of the problem can be approximated by shape functions in the local coordinate η, but, here, it is not necessary to guarantee a continuous approximation across the elements; sometimes it is even necessary to simulate discontinuities, e.g., when one element is loaded by constant tractions and the next is unloaded. Hence, besides the above introduced linear and quadratic shape functions (4.3) and (4.5), respectively, a single middle node x_i^{1e} and the constant shape function ($n = 1$)

$$N_1^1(\eta) = 1 \tag{4.7}$$

can be used to approximate a boundary state $\Phi(\mathbf{x})$.

This means that a boundary state $\Phi(x_i)$ may be approximated in a boundary element Γ_e as

$$\Phi(x_i^e(\eta)) = \Phi^e(\eta) = \sum_{l=1}^{n} \Phi^{le} N_l^n(\eta) \text{ with } 0 \leq \eta \leq 1. \tag{4.8}$$

where n indicates the number of applied nodes per element.

4.1.2 On boundary surfaces in R^3

Again, the first step is to divide the boundary surface Γ into m, in general, curved either quadrilateral or triangular surface elements Γ^e by introducing on the boundary surface node points $\mathbf{x}^{le}$ ($l = 1, ..., n$) where l defines the local numbering of the nodes on the element. The position of a point on Γ^e is expressed by

$$\mathbf{x}^e(\eta_1, \eta_2) = \sum_{l=1}^{n} \mathbf{x}^{le} N_l^n(\eta_1, \eta_2) \tag{4.9}$$

with the given nodal position vectors $\mathbf{x}^{le}$ multiplied by appropriate shape functions $N_l^n(\eta_1, \eta_2)$ with local coordinates (η_1, η_2) lying in the range $(-1, 1)$ or $(0, 1)$. Every shape function has unit value at its associated node and zero value at all other nodes.

Remark: These shape functions have been developed in the Finite Element Method and taken over into the Boundary Element Method.

The parameters (η_1, η_2) define a plane and the curved element is thus mapped, for *quadrilateral elements* onto a square in this plane. An element represented *linearly* in each of the local coordinates η_1 and η_2 is specified by four given nodal values $\mathbf{x}^{le}$ ($l = 1, ..., 4$) each of which has an associated shape function $N_l^4(\eta_1, \eta_2)$. The following Table shows the shape functions for the four node so-called serendipity element in the range $(-1, 1)$:

$\mathbf{x}^{le}$	(η_1, η_2)	$N_l^4(\eta_1, \eta_2)$
$\mathbf{x}^{1e}$	$(1, 1)$	$(1 + \eta_1)(1 + \eta_2)/4$
$\mathbf{x}^{2e}$	$(-1, 1)$	$(1 - \eta_1)(1 + \eta_2)/4$
$\mathbf{x}^{3e}$	$(-1, -1)$	$(1 - \eta_1)(1 - \eta_2)/4$
$\mathbf{x}^{4e}$	$(1, -1)$	$(1 + \eta_1)(1 - \eta_2)/4$

(4.10)

It can easily be seen that, e.g., $N_1^4(\eta_1, \eta_2)$ is 1 when $\eta_1 = 1$ and $\eta_2 = 1$, that is at node $\mathbf{x}^{1e}$, while $N_1^4(\eta_1, \eta_2) = 0$ when either $\eta_1 = -1$ or $\eta_2 = -1$, and hence it is zero at nodes $\mathbf{x}^{2e}$, $\mathbf{x}^{3e}$, and $\mathbf{x}^{4e}$.

When the variation with respect to each of the local coordinates shall be quadratic, one needs eight nodes, i.e.,

$$\mathbf{x}^e(\eta_1,\eta_2) = \sum_{l=1}^{8} \mathbf{x}^{le} N_l^8(\eta_1,\eta_2) \tag{4.11}$$

and the following shape functions in the range $(-1,1)$

$$\begin{array}{lll}
\mathbf{x}^{le} & (\eta_1,\eta_2) & N_l^8(\eta_1,\eta_2) \\
\mathbf{x}^{1e} & (1,1) & (1+\eta_1)(1+\eta_2)(\eta_1+\eta_2-1)/4 \\
\mathbf{x}^{2e} & (-1,1) & (1-\eta_1)(1+\eta_2)(\eta_1-\eta_2-1)/4 \\
\mathbf{x}^{3e} & (-1,-1) & -(1-\eta_1)(1-\eta_2)(\eta_1+\eta_2+1)/4 \\
\mathbf{x}^{4e} & (1,-1) & -(1+\eta_1)(1-\eta_2)(\eta_1-\eta_2+1)/4 \\
\mathbf{x}^{5e} & (1,0) & (1+\eta_1)(1-\eta_2^2)/2 \\
\mathbf{x}^{6e} & (0,1) & (1-\eta_1^2)(1+\eta_2)/2 \\
\mathbf{x}^{7e} & (-1,0) & (1-\eta_1)(1-\eta_2^2)/2 \\
\mathbf{x}^{8e} & (0,-1) & (1-\eta_1^2)(1-\eta_2)/2
\end{array} \tag{4.12}$$

Checks confirm the wanted shape function properties, that is of having unit value at their 'own' node and zero at other nodes. e.g., the shape function $N_3^8(\eta_1,\eta_2)$ associated with the node $\mathbf{x}^{3e}$ gives at $(-1,-1)$

$$N_3^8(-1,-1) = -(1+1)(1+1)(-1-1+1)/4 = 1$$

and evaluated at $(1,1)$

$$N_3^8(1,1) = -(1-1)(1-1)(1+1+1)/4 = 0$$

Similarly, the values of N_3^8 at the remaining nodes are all zero.

An approximation using *triangular elements* and being *linear* in each of the local coordinates η_1 and η_2 needs only three nodes, the three corner nodes of the triangle

$$\mathbf{x}^e(\eta_1,\eta_2) = \sum_{l=1}^{3} \mathbf{x}^{le} N_l^3(\eta_1,\eta_2) \tag{4.13}$$

where the shape functions associated with these corner nodes are dependent on the range $(-1,+1)$ or $(0,+1)$ of the local coordinates (η_1,η_2) defined as

$$\begin{array}{lll}
\mathbf{x}^{le} & (\eta_1,\eta_2) & N_l^3(\eta_1,\eta_2) \\
\mathbf{x}^{1e} & (1,-1) & (1+\eta_1)/2 \\
\mathbf{x}^{2e} & (-1,1) & (1+\eta_2)/2 \\
\mathbf{x}^{3e} & (-1,-1) & -(\eta_1+\eta_2)/2
\end{array}
\quad \text{or} \quad
\begin{array}{lll}
\mathbf{x}^{le} & (\eta_1,\eta_2) & N_l^3(\eta_1,\eta_2) \\
\mathbf{x}^{1e} & (1,0) & \eta_1 \\
\mathbf{x}^{2e} & (0,1) & \eta_2 \\
\mathbf{x}^{3e} & (0,0) & 1-\eta_1-\eta_2
\end{array} \tag{4.14}$$

Obviously, this simple three node triangular element produces planar approximations.

For curved surfaces, six node triangular elements with the associated *quadratic* shape functions $N_l^6(\eta_1, \eta_2)$, $(l = 1, 2, ..., 6)$ on the range $(-1, +1)$

$$\begin{array}{lll} \mathbf{x}^{le} & (\eta_1, \eta_2) & N_l^6(\eta_1, \eta_2) \\ \mathbf{x}^{1e} & (1, -1) & \eta_1(1+\eta_1)/2 \\ \mathbf{x}^{2e} & (-1, 1) & \eta_2(1+\eta_2)/2 \\ \mathbf{x}^{3e} & (-1, -1) & (\eta_1+\eta_2)(\eta_1+\eta_2+1)/2 \\ \mathbf{x}^{4e} & (0, 0) & (1+\eta_1)(1+\eta_2) \\ \mathbf{x}^{5e} & (-1, 0) & -(1+\eta_2)(\eta_1+\eta_2) \\ \mathbf{x}^{6e} & (0, -1) & -(\eta_1+1)(\eta_1+\eta_2) \end{array} \tag{4.15}$$

are more appropriate.

In the same way as the element surface, the **boundary states** of the problem can be approximated by shape functions in the local coordinates η_1 and η_2, but, here, it is not necessary to guarantee a continuous approximation across the elements; sometimes it is even necessary to simulate *discontinuities*, e.g., when one element is loaded by constant tractions and the next is unloaded. Hence, besides the above introduced bilinear and biquadratic shape functions (4.10) and (4.12), respectively, for quadrilateral elements and (4.14) and (4.15), respectively, for triangular elements, a single middle node x_i^{1e} and the constant shape function $(n = 1)$

$$N_1^1(\eta_1, \eta_2) = 1 \tag{4.16}$$

can be used to approximate a boundary state $\Phi(\mathbf{x})$.

Remark: It should be mentioned that, as in the Finite Element Method, also higher order *discontinuous elements* are possible, e.g., a four-node discontinuous quadrilateral element with the shape functions (for more details, see [9])

$$\begin{array}{lll} \mathbf{x}^{le} & (\eta_1, \eta_2) & N_l^4(\eta_1, \eta_2) \\ \mathbf{x}^{1e} & (\frac{1}{2}, \frac{1}{2}) & (1+2\eta_1)(1+2\eta_2)/4 \\ \mathbf{x}^{2e} & (-\frac{1}{2}, \frac{1}{2}) & (1-2\eta_1)(1+2\eta_2)/4 \\ \mathbf{x}^{3e} & (-\frac{1}{2}, -\frac{1}{2}) & (1-2\eta_1)(1-2\eta_2)/4 \\ \mathbf{x}^{4e} & (\frac{1}{2}, -\frac{1}{2}) & (1+2\eta_1)(1-2\eta_2)/4 \end{array} \tag{4.17}$$

Hence, a boundary state $\Phi(x_i)$ may be approximated in a boundary element Γ^e as

$$\Phi(x_i^e(\eta_1, \eta_2)) = \Phi^e(\eta_1, \eta_2) = \sum_{l=1}^{n} \Phi^{le} N_l^n(\eta_1, \eta_2) \tag{4.18}$$

where n indicates the number of applied nodes.

4.2 Integration over boundary elements

After the first step, the discretization of the boundary in boundary elements, the integrals over the whole boundary Γ are decomposed into a sum of m integrals over single elements

Γ^e. As a consequence of the next steps, the elementwise approximation of the boundary geometry and of the boundary states, the integration over the boundary elements has to take into account the actually applied approximation schemes. This will be described in the following sections.

4.2.1 Elements on boundary curves

When integrating along the boundary curve Γ, in each boundary element Γ^e, the mapping between the global physical coordinates $\mathbf{x}$ and the local coordinate η has to be taken into account by

$$d\Gamma_{\mathbf{x}} = \sqrt{(\frac{dx_1^e}{d\eta})^2 + (\frac{dx_2^e}{d\eta})^2} d\eta = \mid J(\eta) \mid d\eta \tag{4.19}$$

where $\mid J(\eta) \mid$ is the so-called *Jacobian*. For a straight boundary element with linear approximation is

$$\frac{dx_i^e}{d\eta} = x_i^{1e}(-1) + x_i^{2e}(+1) = x_i^{2e} - x_i^{1e} \tag{4.20}$$

which results with the element length l_e

$$\mid J(\eta) \mid = \sqrt{(x_1^{2e} - x_1^{1e})^2 + (x_2^{2e} - x_2^{1e})^2} = l_e$$

so that in this case

$$d\Gamma_{\mathbf{x}} = l_e d\eta \tag{4.21}$$

For a parabolic boundary element with quadratic boundary approximation (4.5) is

$$\frac{dx_i^e}{d\eta} = x_i^{1e}(-3 + 4\eta) + x_i^{2e}(4 - 8\eta) + x_i^{3e}(-1 + 4\eta) \tag{4.22}$$

and, hence, the Jacobian $\mid J(\eta) \mid$ (see, (4.19)) becomes rather complicated and analytical integration are rarely possible.

When a boundary element Γ^e of a two-dimensional domain is **circular** with a curvature radius R^e and the coordinates' center of the circumference x_{i0}^e, the coordinates of a generic point along the element can be expressed by means of these 3 parameters as a function of the angle φ (between the x_1-axis and the radial vector $\mathbf{x}^e - \mathbf{x}_0^e$) in the form

$$x_1^e = R^e \cos\varphi + x_{10}^e \tag{4.23}$$
$$x_2^e = R^e \sin\varphi + x_{20}^e \tag{4.24}$$

Representing the angle φ by quadratic shape functions in the local coordinate η

$$\varphi(\eta) = N_1^3(\eta)\varphi_1^e + N_2^3(\eta)\varphi_2^e + N_3^3(\eta)\varphi_3^e \tag{4.25}$$

where φ_1, φ_2,and φ_3 is the angle of the initial node, of an inner node, and of the end node, respectively, on the circular element, gives the quadratic expression

$$\varphi(\eta) = D_3^e\eta^2 + E_3^e\eta + F_3^e \tag{4.26}$$

Dependent on the definition range of η, these coefficients are

$$D_3^e = 2(\varphi_1^e - 2\varphi_2^e + \varphi_3^e),\ E_3^e = -3\varphi_1^e + 4\varphi_2^e - \varphi_3^e,\ F_3^e = \varphi_1^e \text{ for } 0 \leq \eta \leq 1 \quad (4.27)$$
$$D_3^e = 0.5(\varphi_1^e - 2\varphi_2^e + \varphi_3^e),\ E_3^e = 0.5(\varphi_3^e - \varphi_1^e),\ F_3^e = \varphi_2^e \quad \text{for } -1 \leq \eta \leq 1 \quad (4.28)$$

The quadratic relation (4.26) between φ and η allows also an explicit definition of η as a function of φ

$$\eta = \begin{cases} \frac{1}{2D_3}\left(-E_3 + \sqrt{E_3^2 - 4D_3(F_3 - \varphi)}\right) & \text{for } D_3 \neq 0 \\ \frac{1}{E_3}(\varphi - F_3) & \text{for } D_3 = 0 \end{cases} \quad (4.29)$$

where $D_3 = 0$ means $\varphi_2 = 0.5(\varphi_1 + \varphi_3)$, i.e., the node with the angle φ_2 has the same distance to the initial node of the circular element with the angle φ_1 and to the end node with the angle φ_3.

The Jacobian of this transformation is calculated via

$$\frac{dx_1^e}{d\eta} = \frac{dx_1^e}{d\varphi}\frac{d\varphi}{d\eta} = -R^e \sin\varphi\,(2D_3^e + E_3^e)$$
$$\frac{dx_2^e}{d\eta} = \frac{dx_2^e}{d\varphi}\frac{d\varphi}{d\eta} = R^e \cos\varphi\,(2D_3^e + E_3^e)$$

to be

$$J(\eta) = \sqrt{(\frac{dx_1^e}{d\eta})^2 + (\frac{dx_2^e}{d\eta})^2} = R^e\,(2D_3^e + E_3^e) \text{ for } D_3 \neq 0 \quad (4.30)$$
$$= R^e E_3^e \text{ for } D_3 = 0 \quad (4.31)$$

Hence, when $D_3 = 0$, the Jacobian is simply equal to the length of the circular element: $J(\eta) = \Delta s^e = R^e(\varphi_3^e - \varphi_1^e)$ for $0 \leq \eta \leq 1$, or to the half of it: $J(\eta) = \frac{1}{2}\Delta s^e = R^e\frac{1}{2}(\varphi_3^e - \varphi_1^e)$ for $-1 \leq \eta \leq 1$.

When the angle φ is represented by linear shape functions in the local coordinate η

$$\varphi(\eta) = N_1^2(\eta)\varphi_1^e + N_2^2(\eta)\varphi_2^e \quad (4.32)$$

where φ_1 and φ_2 is the angle between the outer normal vector at the initial node, at an inner node, and at the end node, respectively, on the circular element and the x_1-axis, this linear expression is

$$\varphi(\eta) = E_2^e\eta + F_2^e \quad (4.33)$$

with

$$E_2^e = \varphi_2^e - \varphi_1^e,\ F_2^e = \varphi_1^e \text{ for } 0 \leq \eta \leq 1$$
$$E_2^e = 0.5(\varphi_2^e - \varphi_1^e),\ F_2^e = 0.5(\varphi_2^e + \varphi_1^e) \quad \text{for } -1 \leq \eta \leq 1$$

From (4.33) one obtains

$$\eta = \frac{\varphi - F_2^e}{E_2^e}$$

and the Jacobian of this linear transformation with

$$\frac{dx_1^e}{d\eta} = \frac{dx_1^e}{d\varphi}\frac{d\varphi}{d\eta} = -R^e \sin\varphi E_2^e, \ \frac{dx_2^e}{d\eta} = \frac{dx_2^e}{d\varphi}\frac{d\varphi}{d\eta} = R^e \cos\varphi E_2^e$$

as

$$J(\eta) = R^e E_2^e \tag{4.34}$$

Hence, as in the case of the quadratic approximation for the choice $D_3 = 0$, the Jacobian is simply equal to the length of the circular element: $J(\eta) = \Delta s^e = R^e(\varphi_2^e - \varphi_1^e)$ for $0 \le \eta \le 1$, or to the half of it: $J(\eta) = \frac{1}{2}\Delta s^e = R^e \frac{1}{2}(\varphi_2^e - \varphi_1^e)$ for $-1 \le \eta \le 1$.

A boundary integral with the kernel $g(\mathbf{x}, \xi)$ and the state variable $\Phi(\mathbf{x})$ over a boundary element Γ^e is then transformed into an integral over the local coordinate η as:

$$\begin{aligned} \int_{\Gamma^e} g(\mathbf{x},\xi)\Phi(\mathbf{x})d\Gamma_{\mathbf{x}} &= \int_{\eta=0}^{1} g(\mathbf{x}^e(\eta),\xi)\sum_{l=1}^{n}\Phi^{le}N_l^n(\eta) \mid J(\eta) \mid d\eta \\ &= \sum_{l=1}^{n}\Phi^{le}\left\{\int_{\eta=0}^{1} g(\mathbf{x}^e(\eta),\xi)N_l^n(\eta) \mid J(\eta) \mid d\eta\right\} \end{aligned} \tag{4.35}$$

For points ξ in a different boundary element $\Gamma^{e'}$, $e' \neq e$, the integrand of (4.35) is regular for all possible kernels $g(\mathbf{x}, \xi)$ so that those integrals can be numerically evaluated by a Gaussian quadrature formula

$$\int_{\eta=0}^{1} g(\mathbf{x}^e(\eta),\xi)N_l^n(\eta) \mid J(\eta) \mid d\eta = \int_{\eta=0}^{1} f_l^{ne}(\eta)d\eta = \sum_p f_l^{ne}(\eta^p)w_p \tag{4.36}$$

The number of Gaussian points with the abscissae η^p and the weights w_p is dependent on the distance of the point ξ from the integration element Γ^e, generally, between 4 and 10 Gaussian points are taken.

Coincides the point ξ with one of the nodes $\mathbf{x}^{le}$ of the considered boundary element Γ^e, most of the integral kernels become either weakly singular or strongly singular, some even hypersingular. Such integrals should be integrated analytically if possible (see Appendix B).

4.2.2 Elements on boundary surfaces

When integrating over a boundary surface Γ, in each boundary element Γ^e, the mapping between the global physical coordinates $\mathbf{x}$ and the local coordinates η_1 and η_2 has to be taken into account by the surface Jacobian which relates the element of area $d\Gamma_{\mathbf{x}}$ on the surface to the element of area $d\eta_1 d\eta_2$ in the local parameter domain

$$d\Gamma_{\mathbf{x}} = \mid J(\eta_1, \eta_2) \mid d\eta_1 d\eta_2 \tag{4.37}$$

The surface will contain lines corresponding to constant values of the local coordinates η_1 and η_2. Moving along the constant η_1-coordinate line from (η_1, η_2) to $(\eta_1, \eta_2 + d\eta_2)$ results in a point $\mathbf{x}(\eta_1, \eta_2 + d\eta_2)$ with the approximative position vector

$$\mathbf{x}(\eta_1, \eta_2 + d\eta_2) = \mathbf{x}(\eta_1, \eta_2) + \frac{\partial \mathbf{x}}{\partial \eta_2} d\eta_2 + O((d\eta_2)^2)$$

and similarly, moving along a constant η_2-coordinate line gives

$$\mathbf{x}(\eta_1 + d\eta_1, \eta_2) = \mathbf{x}(\eta_1, \eta_2) + \frac{\partial \mathbf{x}}{\partial \eta_1} d\eta_1 + O((d\eta_1)^2)$$

Hence, the vectors along the sides of the surface element of area $d\Gamma_{\mathbf{x}}$ are

$$\begin{aligned} \mathbf{x}(\eta_1, \eta_2 + d\eta_2) - \mathbf{x}(\eta_1, \eta_2) &= \frac{\partial \mathbf{x}}{\partial \eta_2} d\eta_2 + O((d\eta_2)^2) \\ \mathbf{x}(\eta_1 + d\eta_1, \eta_2) - \mathbf{x}(\eta_1, \eta_2) &= \frac{\partial \mathbf{x}}{\partial \eta_1} d\eta_1 + O((d\eta_1)^2) \end{aligned}$$

To the first order, the surface area will be plane and $d\Gamma_{\mathbf{x}}$ is thus given by the formula from vector calculus relating the area of a plane rectangle to the cross product of the vectors on its sides:

$$d\Gamma_{\mathbf{x}} = \left| \frac{\partial \mathbf{x}}{\partial \eta_1} d\eta_1 \times \frac{\partial \mathbf{x}}{\partial \eta_2} d\eta_2 \right| = \left| \frac{\partial \mathbf{x}}{\partial \eta_1} \times \frac{\partial \mathbf{x}}{\partial \eta_2} \right| d\eta_1 d\eta_2 \qquad (4.38)$$

By comparison with (4.37), the surface Jacobian is

$$| J(\eta_1, \eta_2) | = \left| \frac{\partial \mathbf{x}}{\partial \eta_1} \times \frac{\partial \mathbf{x}}{\partial \eta_2} \right| \qquad (4.39)$$

Since the cross product may be expanded as (see (2.11)

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial \eta_1} \times \frac{\partial \mathbf{x}}{\partial \eta_2} &= \hat{\mathbf{e}}_1 \begin{vmatrix} \frac{\partial x_2}{\partial \eta_1} & \frac{\partial x_3}{\partial \eta_1} \\ \frac{\partial x_2}{\partial \eta_2} & \frac{\partial x_3}{\partial \eta_2} \end{vmatrix} - \hat{\mathbf{e}}_2 \begin{vmatrix} \frac{\partial x_1}{\partial \eta_1} & \frac{\partial x_3}{\partial \eta_1} \\ \frac{\partial x_1}{\partial \eta_2} & \frac{\partial x_3}{\partial \eta_2} \end{vmatrix} + \hat{\mathbf{e}}_3 \begin{vmatrix} \frac{\partial x_1}{\partial \eta_1} & \frac{\partial x_2}{\partial \eta_1} \\ \frac{\partial x_1}{\partial \eta_2} & \frac{\partial x_2}{\partial \eta_2} \end{vmatrix} \\ &= \hat{\mathbf{e}}_1 m_{11} + \hat{\mathbf{e}}_2 m_{12} + \hat{\mathbf{e}}_3 m_{13} \end{aligned}$$

where m_{ij} are the minors of $\frac{\partial \mathbf{x}}{\partial \eta_1} \times \frac{\partial \mathbf{x}}{\partial \eta_2}$. Thus

$$| J(\eta_1, \eta_2) | = \left| \frac{\partial \mathbf{x}}{\partial \eta_1} \times \frac{\partial \mathbf{x}}{\partial \eta_2} \right| = \sqrt{m_{11}^2 + m_{12}^2 + m_{13}^2} \qquad (4.40)$$

A boundary integral with the kernel $g(\mathbf{x}, \xi)$ and the state variable $\Phi(\mathbf{x})$ over a boundary element Γ^e is then transformed into an integral over the local coordinates η_1 and η_2, e.g., with the range $-1 \leq \eta_1, \eta_2 \leq 1$ as:

$$\begin{aligned} \int_{\Gamma^e} g(\mathbf{x}, \xi) \Phi(\mathbf{x}) d\Gamma_{\mathbf{x}} &= \int_{\eta_1=-1}^{1} \int_{\eta_2=-1}^{1} g(\mathbf{x}^e(\eta_1, \eta_2), \xi) \sum_{l=1}^{n} \Phi^{le} N_l^n(\eta_1, \eta_2) \mid J(\eta_1, \eta_2) \mid d\eta_1 d\eta_2 \\ &= \sum_{l=1}^{n} \Phi^{le} \left\{ \int_{\eta_1=-1}^{1} \int_{\eta_2=-1}^{1} g(\mathbf{x}^e(\eta_1, \eta_2), \xi) N_l^n(\eta_1, \eta_2) \mid J(\eta_1, \eta_2) \mid d\eta_1 d\eta_2 \right\} \end{aligned} \qquad (4.41)$$

For points ξ in a different boundary element $\Gamma^{e'}$, $e' \neq e$, the integrand of (4.41) is regular for all possible kernels $g(\mathbf{x}, \xi)$ so that those integrals can be numerically evaluated by a Gaussian quadrature formula

$$\begin{aligned} \int_{\eta_1=-1}^{1} \int_{\eta_2=-1}^{1} g(\mathbf{x}^e(\eta_1,\eta_2),\xi) N_l^n(\eta_1,\eta_2) \quad | \quad J(\eta_1,\eta_2) \mid d\eta_1 d\eta_2 &= \int_{\eta_1=-1}^{1} \int_{\eta_2=-1}^{1} f_l^{ne}(\eta_1,\eta_2) d\eta_1 d\eta_2 \\ &= \sum_p \sum_q f_l^{ne}(a_p, a_q) w_p w_q \end{aligned} \tag{4.42}$$

The number of Gaussian points with the abscissae a_p and the weights w_p as well as with the abscissae a_q and the weights w_q is dependent on the distance of the point ξ from the integration element Γ^e, generally, between 4 and 10 Gaussian points are taken.

Coincides the point ξ with one of the nodes $\mathbf{x}^{le}$ of the considered boundary element Γ^e, most of the integral kernels become either weakly singular or strongly singular, some even hypersingular. Such integrals should be integrated analytically if possible (see Appendix B).

4.3 Boundary element equations by point collocation

A 'direct' integral equation, e.g., in the case of a scalar boundary value problem (see, e.g., (3.158))

$$c(\xi)\Phi(\xi) + \int_\Gamma g(\mathbf{x},\xi)\Phi(\mathbf{x}) d\Gamma_{\mathbf{x}} = \int_\Gamma h(\mathbf{x},\xi)\Psi(\mathbf{x}) d\Gamma_{\mathbf{x}} \tag{4.43}$$

with a weakly singular and a strongly singular kernel $h(\mathbf{x}, \xi)$ and $g(\mathbf{x}, \xi)$, respectively, will be transformed by partitioning the boundary Γ into elements Γ_e $(e = 1, ..., m)$, i.e., decomposing the integrals over the whole boundary Γ into a sum of m integrals over single elements Γ_e

$$\int_\Gamma [....] \, d\Gamma_{\mathbf{x}} = \sum_{e=1}^{m} \int_{\Gamma_e} [....] \, d\Gamma_{\mathbf{x}} \tag{4.44}$$

and by the approximation of the boundary, i.e., of the boundary coordinates x_i^e (see, (4.1)), and of the boundary states $\Phi(\mathbf{x})$ and $\Psi(\mathbf{x})$ (see (4.8)), as it is shown in (4.35)), into the following approximate equation

$$\begin{aligned} c(\xi)\Phi(\xi) &= -\sum_{e=1}^{m} \sum_{l=1}^{n} \Phi^{le} \left\{ \int_{\eta=0}^{1} g(\mathbf{x}^e(\eta),\xi) N_l^n(\eta) \mid J(\eta) \mid d\eta \right\} \\ &\quad + \sum_{e=1}^{m} \sum_{l=1}^{n} \Psi^{le} \left\{ \int_{\eta=0}^{1} h(\mathbf{x}^e(\eta),\xi) N_l^n(\eta) \mid J(\eta) \mid d\eta \right\} \end{aligned} \tag{4.45}$$

Remark: In many physical boundary value problems, the state variable $\Phi(\mathbf{x})$ is continous along the whole boundary (e.g., the temperature in the case of heat conduction problems) while the state $\Psi(\mathbf{x})$ is discontinuous since it is often related to the normal vector which jumps at corner points. Such discontinuities have to be taken into account in approximating this state.

The number of unknown node values Φ^{le} or Ψ^{le} of the shape functions of $\Phi(\mathbf{x})$ and $\Psi(\mathbf{x})$, respectively, in this single equation (4.7) is dependent on the grade $(n-1)$ of the applied shape functions: - for constant shape functions, i.e., $n = 1$, as well as for linear shape functions, i.e., $n = 2$, one has m unknowns in the case of a closed smooth contour (i.e., in the linear case assuming along the whole boundary a continous approximation) and $2m$ unknowns for quadratic shape functions, i.e., $n = 3$, (again assuming a continous approximation). Hence, one need m and $2m$, respectively, equations for their unique determination.

The most simple and mostly used way is to evaluate this equation (4.45) at as many points $\xi = \mathbf{x}^j$ as one needs where it is advisable and useful to choose for this purpose, i.e., as these *collocation points,* the node points $\mathbf{x}^{le}$ of the shape functions:

$$\begin{aligned} c(\mathbf{x}^j)\Phi(\mathbf{x}^j) &= -\sum_{e=1}^{m}\sum_{l=1}^{n}\Phi^{le}\left\{\int_{\eta=0}^{1} g(\mathbf{x}^e(\eta),\mathbf{x}^j)N_l^n(\eta) \mid J(\eta) \mid d\eta\right\} \\ &\quad +\sum_{e=1}^{m}\sum_{l=1}^{n}\Psi^{le}\left\{\int_{\eta=0}^{1} h(\mathbf{x}^e(\eta),\mathbf{x}^j)N_l^n(\eta) \mid J(\eta) \mid d\eta\right\} \end{aligned} \tag{4.46}$$

Since for linear and all higher grade approximations ($n \geqq 2$) the end node of an element Γ_e is also the initial node of the sequent element Γ_{e+1}, i.e., $\mathbf{x}^{ne} = \mathbf{x}^{1e+1}$, it is helpful to introduce a global node numbering $\mathbf{x}^j, j = 1, 2, ..., N$, as follows

$$\begin{aligned} \text{for } n &= 2, \text{ i.e. } N = m: \ \ \mathbf{x}^1 = \mathbf{x}^{11} = \mathbf{x}^{2m},\ \mathbf{x}^2 = \mathbf{x}^{21} = \mathbf{x}^{12},\ \mathbf{x}^3 = \mathbf{x}^{22} = \mathbf{x}^{13},\ ... \\ \text{for } n &= 3, \text{ i.e. } N = 2m: \ \ \mathbf{x}^1 = \mathbf{x}^{11} = \mathbf{x}^{3m},\ \mathbf{x}^2 = \mathbf{x}^{21},\ \mathbf{x}^3 = \mathbf{x}^{31} = \mathbf{x}^{12},\ ... \end{aligned}$$

4.3.1 Approximation by constant shape functions

Physically not realistic but most simple is an approximation of the state functions by constant shape functions ($n = 1$) and collocation at the middle node $\mathbf{x}^{1e} = \mathbf{x}^e$ of each element Γ_e.

With $N_1^1(\eta) = 1$ one has only one representative value $\Phi^{1e} = \Phi^e$ and $\Psi^{1e} = \Psi^e$ for the boundary states in each element and (4.46) yields with a linear approximation of the boundary Γ (i.e., $\mid J(\eta) \mid = l_e$)

$$\frac{1}{2}\Phi^j = \frac{1}{2}\Phi(\mathbf{x}^j) = -\sum_{e=1}^{m}\Phi^e\left\{\int_{\eta=0}^{1} g(\mathbf{x}^e(\eta),\mathbf{x}^j)l_e\,d\eta\right\} + \sum_{e=1}^{m}\Psi^e\left\{\int_{\eta=0}^{1} h(\mathbf{x}^e(\eta),\mathbf{x}^j)l_e\,d\eta\right\} \tag{4.47}$$

where, due to the collocation at the middle nodes $\mathbf{x}^j$ of the straight and, therefore, smooth boundary elements, the factor $c(\mathbf{x}^j)$ is always $c(\mathbf{x}^j) = 1/2$. Finally, one obains with the matrices

$$G_{je} = \int_{\eta=0}^{1} g(\mathbf{x}^e(\eta),\mathbf{x}^j)l_e\,d\eta \ \text{ and } \ H_{je} = \int_{\eta=0}^{1} h(\mathbf{x}^e(\eta),\mathbf{x}^j)l_e\,d\eta \tag{4.48}$$

the following system of m equations for m unknown boundary values

$$\sum_{e=1}^{m}\left(\frac{1}{2}\delta_{je} + G_{je}\right)\Phi^e = \sum_{e=1}^{m} H_{je}\Psi^e \quad \text{for } j = 1, 2, ..., m \tag{4.49}$$

4.3.1.1 Example: Stationary heat conduction

In the case of a stationary heat conduction problem, the boundary states in (4.49) are the temperature Θ and the heat flux q_n, i.e., $\Phi^e \triangleq \Theta^e$ and $\Psi^e \triangleq q_n^e$, and the kernels $g(\mathbf{x}^e(\eta), \xi) \triangleq -q_n^*(\mathbf{x}, \xi)/\lambda_0 = \frac{-1}{2\pi} r_{,n}/r$ and $h(\mathbf{x}^e(\eta), \mathbf{x}^j) \triangleq -\Theta^*(\mathbf{x}, \xi)/\lambda_0 = \frac{-1}{2\pi\lambda_0} \ln(r/c)$ (see, (3.152) and (3.148)). Hence, one has to evaluate the following expressions for the matrices G_{je} and H_{je}, respectively, ($\mathbf{r}^{1ej} = \mathbf{x}^{1e} - \mathbf{x}^j$):

$$
\begin{aligned}
G_{je} &= \frac{-1}{2\pi} \int_{\eta=0}^{1} \frac{1}{r_{je}(\eta)} \frac{\partial r_{je}(\eta)}{\partial x_i} n_i^e l_e d\eta = \frac{l_e n_i^e}{2\pi} \int_{\eta=0}^{1} \frac{x_i^{1e}(1-\eta) + x_i^{2e}\eta - x_i^j}{r_{je}^2(\eta)} d\eta \\
&= \frac{-l_e}{2\pi} \int_{\eta=0}^{1} \frac{n_i^e(x_i^{1e} - x_i^j) + \eta(x_i^{2e} - x_i^{1e}) n_i^e}{r_{je}^2(\eta)} d\eta \qquad (4.50) \\
&= \frac{-l_e}{2\pi} \mathbf{n}^e \cdot \mathbf{r}^{1ej} \int_{\eta=0}^{1} \frac{d\eta}{l_e^2 \eta^2 + 2\mathbf{r}^{1ej} \cdot \mathbf{t}^e l_e \eta + \mid \mathbf{r}^{1ej} \mid^2} \qquad (4.51)
\end{aligned}
$$

$$
\begin{aligned}
H_{je} &= \frac{-l_e}{4\pi\lambda_0} \int_{\eta=0}^{1} \ln\left(\frac{r_{je}^2(\eta)}{l_0^2}\right) d\eta \\
&= \frac{-l_e}{4\pi\lambda_0} \int_{\eta=0}^{1} \ln\left(\frac{l_e^2\eta^2 + 2\mathbf{r}^{1ej} \cdot \mathbf{t}^e l_e \eta + \mid \mathbf{r}^{1ej} \mid^2}{l_0^2}\right) d\eta \\
&= \frac{-l_e}{4\pi\lambda_0} \int_{\eta=0}^{1} \left\{ \ln\left(\eta^2 + \frac{2}{l_e} \mathbf{r}^{1ej} \cdot \mathbf{t}^e \eta + \frac{\mid \mathbf{r}^{1ej} \mid^2}{l_e^2}\right) + \ln\left(\frac{l_e^2}{l_0^2}\right) \right\} d\eta \qquad (4.52)
\end{aligned}
$$

where the constant reference value c in Θ^* has been taken as the length of the shortest element Γ_e, i.e., $l_0 = \min\{l_e \,|\, e = 1, ..., m\}$ and, moreover, the following relations hold for a linear boundary approximation

$$
\begin{aligned}
r_{je}^2(\eta) &= \left[x_1^{1e}(1-\eta) + x_1^{2e}\eta - x_1^j\right]^2 + \left[x_2^{1e}(1-\eta) + x_2^{2e}\eta - x_2^j\right]^2 \\
&= l_e^2\eta^2 + 2(x_i^{1e} - x_i^j) l_e t_i^e \eta + (x_i^{1e} - x_i^j)(x_i^{1e} - x_i^j) \\
&= l_e^2\eta^2 + 2\mathbf{r}^{1ej} \cdot \mathbf{t}^e l_e \eta + \mid \mathbf{r}^{1ej} \mid^2 \qquad (4.53)
\end{aligned}
$$

$$
(x_i^{2e} - x_i^{1e}) = l_e t_i^e \rightarrow (x_i^{2e} - x_i^{1e}) n_i^e = l_e t_i^e n_i^e = 0 \qquad (4.54)
$$

For the **singular** elements, i.e., for a collocation at the middle node $x_i^j = 0,5(x_i^{2e} + x_i^{1e})$ of the considered element Γ_e, one obtains

$$
\mathbf{r}^{1ej} \triangleq x_i^{1e} - x_i^j = -0,5(x_i^{2e} - x_i^{1e}) = -0,5 l_e t_i^e
$$

so that

$$
\mathbf{n}^e \cdot \mathbf{r}^{1ej} \triangleq n_i^e(x_i^{1e} - x_i^j) = -0,5 l_e n_i^e t_i^e = 0
$$

and, hence, one obtains from (4.50) regarding also (4.54)

$$
G_{ee} = 0.
$$

The reason is $\partial r/\partial n = 0$ for this middle node collocation point.

For a collocation at this middle node is due to

$$\begin{aligned} \mathbf{t}^e \cdot \mathbf{r}^{1ej} &\triangleq t_i^e(x_i^{1e} - x_i^j) = -0,5 l_e t_i^e t_i^e = -0,5 l_e \\ \mathbf{r}^{1ej} \cdot \mathbf{r}^{1ej} &\triangleq (x_i^{1e} - x_i^j)(x_i^{1e} - x_i^j) = 0.25 l_e^2 \end{aligned}$$

the respective singular element in the H-matrix (for details about the evaluation of this improper integral, see Appendix B, 4.5.1)

$$\begin{aligned} H_{ee} &= \frac{-l_e}{4\pi\lambda_0} \int\limits_{\eta=0}^{1} \left\{ \ln \left(\eta - \frac{1}{2} \right)^2 + \ln \left(\frac{l_e^2}{l_0^2} \right) \right\} d\eta \\ &= \frac{-l_e}{4\pi\lambda_0} \lim_{\varepsilon \to 0} \left[\int\limits_{\eta=0}^{\frac{1}{2}-\varepsilon} \ln \left(\eta - \frac{1}{2} \right)^2 d\eta + \int\limits_{\eta=\frac{1}{2}+\varepsilon}^{1} \ln \left(\eta - \frac{1}{2} \right)^2 d\eta + \ln \left(\frac{l_e^2}{l_0^2} \right) \right] \\ &= \frac{l_e}{2\pi\lambda_0} [1 + \ln(2) - \ln(l_e/l_0)] \end{aligned} \tag{4.55}$$

For all other collocation points (where $n_i^e(x_i^{1e} - x_i^j) = \mathbf{n}^e \cdot \mathbf{r}^{1ej} \neq 0$) holds

$$\begin{aligned} G_{je} &= \frac{-l_e}{2\pi} \mathbf{n}^e \cdot \mathbf{r}^{1ej} \left[\frac{2}{2l_e \mid \mathbf{n}^e \cdot \mathbf{r}^{1ej} \mid} \arctan \frac{l_e \eta + \mathbf{t}^e \cdot \mathbf{r}^{1ej}}{\mid \mathbf{n}^e \cdot \mathbf{r}^{1ej} \mid} \right]_{\eta=0}^{\eta=1} \\ &= \frac{-1}{2\pi} \operatorname{sign}\left(\mathbf{n}^e \cdot \mathbf{r}^{1ej}\right) \left\{ \arctan \frac{l_e + \mathbf{t}^e \cdot \mathbf{r}^{1ej}}{\mid \mathbf{n}^e \cdot \mathbf{r}^{1ej} \mid} - \arctan \frac{\mathbf{t}^e \cdot \mathbf{r}^{1ej}}{\mid \mathbf{n}^e \cdot \mathbf{r}^{1ej} \mid} \right\} \end{aligned} \tag{4.56}$$

$$H_{je} = \frac{-1}{4\pi\lambda_0} \left\{ \begin{array}{c} (l_e + \mathbf{t}^e \cdot \mathbf{r}^{1ej}) \ln \left(1 + \frac{2}{l_e} \mathbf{t}^e \cdot \mathbf{r}^{1ej} + \frac{|\mathbf{r}^{1ej}|^2}{l_e^2} \right) \\ -\mathbf{t}^e \cdot \mathbf{r}^{1ej} \ln \left(\frac{|\mathbf{r}^{1ej}|^2}{l_e^2} \right) - 2l_e + l_e \ln \left(\frac{l_e^2}{l_0^2} \right) \\ +2 \mid \mathbf{n}^e \cdot \mathbf{r}^{1ej} \mid \left[\arctan \frac{l_e + \mathbf{t}^e \cdot \mathbf{r}^{1ej}}{|\mathbf{n}^e \cdot \mathbf{r}^{1ej}|} - \arctan \frac{\mathbf{t}^e \cdot \mathbf{r}^{1ej}}{|\mathbf{n}^e \cdot \mathbf{r}^{1ej}|} \right] \end{array} \right\} \tag{4.57}$$

Remarks:

a) There holds the following separation of components for length of the distance vector between the collocation point x_i^j and the initial node x_i^{1e} of the element Γ_e

$$\left| \mathbf{r}^{1ej} \right|^2 = (x_i^{1e} - x_i^j)(x_i^{1e} - x_i^j) = \left[n_i^e (x_i^{1e} - x_i^j) \right]^2 + \left[t_i^e (x_i^{1e} - x_i^j) \right]^2$$

where n_i^e and t_i^e means the unit normal vector and unit tangential vecor, respectively, in the straight element Γ_e.

b) In the above evaluation of G_{je}, the following integral was applied (for $4b - a^2 > 0$):

$$\int \ln(\eta^2 + a\eta + b)\, d\eta = (\eta + \frac{a}{2}) \ln(\eta^2 + a\eta + b) - 2\eta + \sqrt{4b - a^2} \arctan \frac{2\eta + a}{\sqrt{4b - a^2}}$$

4.3.1.2 Exercise 14: Stationary heat conduction in a rectangular domain-constant shape functions

In a rectangular domain $\Omega = \{(x_1, x_2) \mid 0 \leq x_1 \leq l_1 = 1m,\ 0 \leq x_2 \leq l_2 = 2m\}$, the temperature is prescribed at two opposite sides $x_2 = 0$ and $x_2 = l_2$ to be $\Theta(x_1, 0) = 0$ and $\Theta(x_1, l_2) = \overline{\Theta} > 0$ while at the other two sides $x_1 = 0$ and $x_1 = l_1$ the heat flux is stopped, i.e., $q_n = 0$. For this problem (see [2]), the exact temperature distribution is known to be $\Theta(x_1, x_2) = \overline{\Theta}\ x_2/l_2$ from which one can determine via (3.146) the heat flux across the boundary to be as $q_n(x_1, x_2) = \lambda_0 n_i(x_1, x_2)\partial\Theta(x_1, x_2)/\partial x_i = \lambda_0 n_2(x_1, x_2)\overline{\Theta}\ /l_2$.

Solve this problem applying the above given boundary element approximation by taken each of the four sides as a boundary element and compare with the known exact solution.

4.3.2 Approximation by linear and higher grade shape functions

Continous state functions $\Phi(\mathbf{x})$ and $\Psi(\mathbf{x})$, which are approximated by shape functions with n ($\geqq 2$) nodes per element, have in successive boundary elements Γ_e und Γ_{e+1} the same node value, i.e., $\Phi^{ne} = \Phi^{1e+1}$ and $\Psi^{ne} = \Psi^{1e+1}$, respectively, and, hence, an equal global node value numbering Φ^i and Ψ^i, respectively. Therefore, for each collcation point $\mathbf{x}^j$, the elements in one line of the matrices $\mathbf{G}$ and $\mathbf{H}$ related to the boundary state node values Φ^i and Ψ^i, respectively, read as:

$$G_{ji} = \begin{cases} \int\limits_{\eta=0}^{1} g(\mathbf{x}^e(\eta), \mathbf{x}^j) N_n^n(\eta) \mid J(\eta) \mid d\eta + \int\limits_{\eta=0}^{1} g(\mathbf{x}^{e+1}(\eta), \mathbf{x}^j) N_1^n(\eta) \mid J(\eta) \mid d\eta \\ \qquad \text{for } \Phi^i = \Phi^{ne} = \Phi^{1e+1} \\ \int\limits_{\eta=0}^{1} g(\mathbf{x}^e(\eta), \mathbf{x}^j) N_l^n(\eta) \mid J(\eta) \mid d\eta \quad \text{for } \Phi^i = \Phi^{le} \ \text{ if } l \neq 1, n \ (n > 2) \end{cases} \tag{4.58}$$

$$H_{ji} = \begin{cases} \int\limits_{\eta=0}^{1} h(\mathbf{x}^e(\eta), \mathbf{x}^j) N_n^n(\eta) \mid J(\eta) \mid d\eta + \int\limits_{\eta=0}^{1} h(\mathbf{x}^{e+1}(\eta), \mathbf{x}^j) N_1^n(\eta) \mid J(\eta) \mid d\eta \\ \qquad \text{for } \Psi^i = \Psi^{ne} = \Psi^{1e+1} \\ \int\limits_{\eta=0}^{1} h(\mathbf{x}^e(\eta), \mathbf{x}^j) N_l^n(\eta) \mid J(\eta) \mid d\eta \quad \text{for } \Psi^i = \Psi^{le} \ \text{ if } l \neq 1, n \ (n > 2) \end{cases} \tag{4.59}$$

With that, for continous state functions on a smooth boundary of a $2d$-body, the following system of equations for the N (for $n = 2$ is $N = m$ and for $n = 3$ is $N = 2m$) unknown node values is given ($c^j = c(\mathbf{x}^j)$ and $\Phi^j = \Phi(\mathbf{x}^j)$)

$$\sum_{i=1}^{N} \left(c^j \delta_{ji} + G_{ji}\right) \Phi^i = \sum_{i=1}^{N} H_{ji} \Psi^i \quad \text{for } j = 1, 2, .., N \tag{4.60}$$

Remark: For a linear approximation of the boundary Γ and linear shape functions in each boundary element Γ_e for a continous state function Φ the elements (4.58) are explicitly:

$$\text{for } i \ \geqq \ 2 : G_{ji} = \int\limits_{\eta=0}^{1} g(\mathbf{x}^e(\eta), \mathbf{x}^j)\eta l_e \, d\eta + \int\limits_{\eta=0}^{1} g(\mathbf{x}^{e+1}(\eta), \mathbf{x}^j)(1-\eta) l_{e+1} \, d\eta \quad \text{with } i = e+1$$

$$\text{for } i \ = \ 1 : G_{j1} = \int\limits_{\eta=0}^{1} g(\mathbf{x}^m(\eta), \mathbf{x}^j)\eta l_m \, d\eta + \int\limits_{\eta=0}^{1} g(\mathbf{x}^1(\eta), \mathbf{x}^j)(1-\eta) l_1 \, d\eta \quad \text{at } \mathbf{x}^1 = \mathbf{x}^{m+1}$$

At corners of the boundary approximation, the direction of the normal vector jumps and, hence, a boundary state, e.g., $\Psi(\mathbf{x})$, which is dependent on the normal vector, is also discontinous there, i.e., $\Psi^{ne} \neq \Psi^{1e+1}$ when the boundary elements Γ_e and Γ_{e+1} form a corner. But also at a point on a smooth boundary, the prescribed boundary state function $\Psi(\mathbf{x})$ itself can be discontinous, e.g., in a heat conduction problem when the permeability and consequently the heat flux changes suddenly. For the case that such discontinuities $\Psi^{ne} \neq \Psi^{1e+1}$ are assumed at the transition points between all elements increase the number of node values Ψ^i, e.g., for $n = 2$ from $N = m$ to $N' = 2m$ and for $n = 3$ from $N = 2m$ to $N' = 3m$:

$$\sum_{i=1}^{m} \left(c^j \delta_{ji} + G_{ji}\right) \Phi^i = \sum_{e=1}^{m} \sum_{l=1}^{n} H_{je}^{l} \Psi^{le} \quad \text{für } j = 1, 2, .., m \tag{4.61}$$

where, different to (4.59), the coefficients of Ψ^{le}, i.e., the elements of $\mathbf{H}$ are defined as

$$\begin{aligned} H_{je}^{l} &= \int_{\eta=0}^{1} h(\mathbf{x}^e(\eta), \mathbf{x}^j) N_l^n(\eta) \mid J(\eta) \mid d\eta \\ \text{with } j &= 1(1)m \text{ and } l = 1(1)n,\ e = 1(1)m\ (n > 2) \end{aligned} \tag{4.62}$$

It should be mentioned that in spite of those discontinuities, $\Psi^{ne} \neq \Psi^{1e+1}$, for the most types of boundary conditions, the system (4.61) obtained by collocation at the nodes of the shape functions allows to determine all unknown nodal values uniquely since:

• at a node on a smooth boundary, a discontinuity of $\Psi(\mathbf{x})$ appears only when it is prescibed, i.e., only the nodal value of the continous state $\Phi(\mathbf{x})$ is unknown at those nodes,

• at a corner node, almost all combinations of boundary conditions 'ahead' and 'behind' a corner produce only one unknown nodal value, e.g.,

-when ahead Ψ^{ne} and behind Ψ^{1e+1} is prescribed: $\Phi^{ne} = \Phi^{1e+1}$ is the single unknown,

- when ahead Ψ^{ne} and behind Φ^{1e+1} $(= \Phi^{ne})$ is prescribed: Ψ^{1e+1} is the single unknown,

- when ahead Φ^{ne} $(= \Phi^{1e+1})$ and behind Ψ^{1e+1} is prescribed: Ψ^{ne} is the single unknown,

Only for the case

- when ahead Φ^{ne} and behind $\Phi^{1e+1} = \Phi^{ne}$ is prescribed, the two values Ψ^{ne} and Ψ^{1e+1} are both unknown and, since the collocation at the corner node delivers only one equation, one equation is missing.

Often, these needed two equations (instead of only one) are produced by using the 'double node concept' (for more details see [6], pp. 87), i.e., two collocation points (instead of the one at the corner) located 'near' the corner in the element ahead and behind, respectively, in the case of linear shape functions often in a distance of $l_e/3$ from the corner.

If the discontinuity at a corner $\mathbf{x} = \mathbf{x}^E$ is only produced by the jump of the normal vector ahead and behind the corner, i.e., $\mathbf{n}^-(\mathbf{x}^E) \neq \mathbf{n}^+(\mathbf{x}^E)$, sometimes a physically based extra condition can be found and the 'double nodes' at the corner can be avoided (see the following example).

4.3.2.1 Example: Stationary heat conduction

In the stationary heat conduction problem, the state $\Psi(\mathbf{x})$ means the heat flux $q_n(\mathbf{x})$ which is defined on the boundary Γ as $q_n(\mathbf{x}) = \lambda_0 n_i(\mathbf{x})\partial\Theta(\mathbf{x})/\partial x_i$ and, hence, is dependent on the normal vector $n_i(\mathbf{x})$. Since the gradient of the temperature is continous also at a corner $\mathbf{x}^E$ of the boundary, the following relations hold for the normal and tangential components of this temperature gradient ahead and behind a corner

$$\frac{\partial\Theta(\mathbf{x})}{\partial\mathbf{n}^-}n_i^- + \frac{\partial\Theta(\mathbf{x})}{\partial\mathbf{t}^-}t_i^- = \frac{\partial\Theta(\mathbf{x})}{\partial x_i} = \frac{\partial\Theta(\mathbf{x})}{\partial\mathbf{n}^+}n_i^+ + \frac{\partial\Theta(\mathbf{x})}{\partial\mathbf{t}^+}t_i^+ \tag{4.63}$$

where $\mathbf{n}^- = (\cos\varphi^-, \sin\varphi^-)$, $\mathbf{t}^- = (-\sin\varphi^-, \cos\varphi^-)$ and $\mathbf{n}^+ = (\cos\varphi^+, \sin\varphi^+)$, $\mathbf{t}^+ = (-\sin\varphi^+, \cos\varphi^+)$ are the normal and tangential unit vectors, respectively, ahead and behind the corner. With that, the above mentioned extra equation (which is only necessary when ahead and behind a corner the temperature is prescribed and, consequently, the tangential derivatives ahead $\partial\Theta(\mathbf{x})/\partial\mathbf{t}^-$ and behind $\partial\Theta(\mathbf{x})/\partial\mathbf{t}^+$ are known) at a corner with internal angle $\Delta\varphi = \pi - \varphi^+ + \varphi^-$ can be given as the following relation between the heat flux $q_n^-(\mathbf{x})$ and $q_n^+(\mathbf{x})$ ahead and behind the corner, respectively:

$$q_n^+(\mathbf{x}) = \lambda_0\frac{\partial\Theta(\mathbf{x})}{\partial\mathbf{n}^+} = \lambda_0\sin\Delta\varphi\frac{\partial\Theta(\mathbf{x})}{\partial\mathbf{t}^-} - \cos\Delta\varphi q_n^-(\mathbf{x}) \tag{4.64}$$

or equivalently

$$q_n^-(\mathbf{x}) = \lambda_0\frac{\partial\Theta(\mathbf{x})}{\partial\mathbf{n}^-} = -\cos\Delta\varphi q_n^+(\mathbf{x}) - \lambda_0\sin\Delta\varphi\frac{\partial\Theta(\mathbf{x})}{\partial\mathbf{t}^+} \tag{4.65}$$

Applying a linear approximation for the boundary and linear shape functions for $\Theta(\mathbf{x})$ and $q_n(\mathbf{x})$, one obtains for heat conduction problems without interior heat sources, i.e., with $W_q(\mathbf{x}) = 0$, from (3.158) the following integral equation for the temperature at a point $\mathbf{x}^j$

$$\begin{aligned} c(\mathbf{x}^j)\Theta(\mathbf{x}^j) &= \sum_{e=1}^{m}\left\{\begin{array}{c} q_n^{1e}\int\limits_{\eta=0}^{1}\Theta^*(\mathbf{x}^e(\eta),\mathbf{x}^j)(1-\eta)l_e\,d\eta \\ +q_n^{2e}\int\limits_{\eta=0}^{1}\Theta^*(\mathbf{x}^e(\eta),\mathbf{x}^j)\eta l_e\,d\eta \end{array}\right\} \\ &\quad -\sum_{e=1}^{m}\left\{\begin{array}{c} \Theta^{1e}\int\limits_{\eta=0}^{1}q_n^*(\mathbf{x}^e(\eta),\mathbf{x}^j)(1-\eta)l_e\,d\eta \\ +\Theta^{2e}\int\limits_{\eta=0}^{1}q_n^*(\mathbf{x}^e(\eta),\mathbf{x}^j)\eta l_e\,d\eta \end{array}\right\} \end{aligned} \tag{4.66}$$

with

$$c(\mathbf{x}^j) = \left\{\begin{array}{cl} \frac{\Delta\varphi(\mathbf{x}^j)}{2\pi} & \text{for boundary points } \mathbf{x}^j \\ 1 & \text{for interior points } \mathbf{x}^j \end{array}\right. \tag{4.67}$$

or corresponding (4.61) in a matrix notation ($\Theta^e = \Theta^{1e}$ with $\Theta^{2e} = \Theta^{1e+1}$)

$$c(\mathbf{x}^j)\Theta(\mathbf{x}^j) = \sum_{e=1}^{m}\left\{H_{je}^1 q_n^{1e} + H_{je}^2 q_n^{2e}\right\} - \sum_{e=1}^{m} G_{je}\Theta^e \quad \text{für } j = 1, 2, .., m \tag{4.68}$$

where $\Delta\varphi(\mathbf{x}^j) = 2\pi - (\varphi_2 - \varphi_1)$ means the internal angle at a boundary point $\mathbf{x}^j$.

Related to the everywhere continous temperture nodal value $\Theta^k = \Theta^{2e} = \Theta^{1e+1}$, $k = e+1$, $e = 1(1)m$ ($\Theta^{m+1} = \Theta^1 = \Theta^{1m+1}$ means that the boundary contour is closed) one obtains from (3.152) explicitly the following expressions for the matrix coefficients G_{jk} ($r_i^{1ej} = x_i^{1e} - x_i^j$):

for $x_i^j = x_i^{1e+1} = x_i^{2e}$ **and each** $k = 1(1)m$ (even when the collocation point x_i^j is at a corner of the linearly approximated boundary):

$$G_{jk} = 0,$$

for $x_i^j \neq x_i^{1e+1} = x_i^{2e}$ **and** $k = e+1 > 1$ **and** $n_i^e(x_i^{1e} - x_i^j) = \mathbf{n}^e \cdot \mathbf{r}^{1ej} \neq 0$ or $\mathbf{n}^{e+1} \cdot \mathbf{r}^{1e+1j} \neq 0$ (for $\mathbf{n}^e \cdot \mathbf{r}^{1ej} = 0$ or $\mathbf{n}^{e+1} \cdot \mathbf{r}^{1e+1j} = 0$, the respective integrals with those factors disappear):

$$\begin{aligned} G_{jk} &= \frac{-l_e}{2\pi}\mathbf{n}^e \cdot \mathbf{r}^{1ej} \int_{\eta=0}^{1} \frac{\eta d\eta}{l_e^2\eta^2 + 2\mathbf{t}^e \cdot \mathbf{r}^{1ej} l_e \eta + \mid \mathbf{r}^{1ej} \mid^2} \\ &\quad - \frac{l_{e+1}}{2\pi}\mathbf{n}^{e+1} \cdot \mathbf{r}^{1e+1j} \int_{\eta=0}^{1} \frac{(1-\eta) d\eta}{l_{e+1}^2\eta^2 + 2\mathbf{t}^{e+1} \cdot \mathbf{r}^{1e+1j} l_{e+1} \eta + \mid \mathbf{r}^{1e+1j} \mid^2} \\ &= \frac{-\mathbf{n}^e \cdot \mathbf{r}^{1ej}}{2\pi l_e} \left\{ \begin{array}{c} \frac{1}{2}\ln\left(1 + 2\frac{l_e \mathbf{t}^e \cdot \mathbf{r}^{ej}}{|\mathbf{r}^{1ej}|^2} + \frac{l_e^2}{|\mathbf{r}^{1ej}|^2}\right) \\ -\frac{\mathbf{t}^e \cdot \mathbf{r}^{ej}}{|\mathbf{n}^e \cdot \mathbf{r}^{ej}|}\left(\arctan\frac{l_e + \mathbf{t}^e \cdot \mathbf{r}^{1ej}}{|\mathbf{n}^e \cdot \mathbf{r}^{1ej}|} - \arctan\frac{\mathbf{t}^e \cdot \mathbf{r}^{1ej}}{|\mathbf{n}^e \cdot \mathbf{r}^{1ej}|}\right) \end{array} \right\} \\ &\quad + \frac{\mathbf{n}^{e+1} \cdot \mathbf{r}^{1e+1j}}{2\pi l_{e+1}} \left\{ \begin{array}{c} \frac{1}{2}\ln\left(1 + 2\frac{l_{e+1} \mathbf{t}^{e+1} \cdot \mathbf{r}^{1e+1j}}{|\mathbf{r}^{1e+1j}|^2} + \frac{l_{e+1}^2}{|\mathbf{r}^{1e+1j}|^2}\right) \\ -\frac{l_{e+1} + \mathbf{t}^{e+1} \cdot \mathbf{r}^{1e+1j}}{|\mathbf{n}^{e+1} \cdot \mathbf{r}^{1e+1j}|}\left(\begin{array}{c}\arctan\frac{l_{e+1} + \mathbf{t}^{e+1} \cdot \mathbf{r}^{1e+1j}}{|\mathbf{n}^{e+1} \cdot \mathbf{r}^{1e+1j}|} \\ - \arctan\frac{\mathbf{t}^{e+1} \cdot \mathbf{r}^{1e+1j}}{|\mathbf{n}^{e+1} \cdot \mathbf{r}^{1e+1j}|}\end{array}\right) \end{array} \right\} \end{aligned} \tag{4.69}$$

for $x_i^j \neq x_i^{11} = x_i^{1m+1} = x_i^{2m}$ **and** $k = 1$ **as well as** $n_i^m(x_i^{1m} - x_i^j) \neq 0$ or $n_i^1(x_i^{11} - x_i^j) \neq 0$ (for $n_i^m(x_i^{1m} - x_i^j) = 0$ or $n_i^1(x_i^{11} - x_i^j) = 0$, the respective integrals with those factors disappear):

$$\begin{aligned} G_{j1} &= \frac{-n_i^m(x_i^{1m} - x_i^j)}{2\pi l_m} \left\{ \begin{array}{c} \frac{1}{2}\ln\left(1 + 2\frac{l_m(x_i^{1m} - x_i^j)t_i^m}{|\mathbf{x}^{1m} - \mathbf{x}^j|^2} + \frac{l_m^2}{|\mathbf{x}^{1m} - \mathbf{x}^j|^2}\right) \\ -\frac{(x_i^{1m} - x_i^j)t_i^m}{|n_i^m(x_i^{1m} - x_i^j)|}\left(\arctan\frac{l_m + (x_i^{1m} - x_i^j)t_i^m}{|n_i^m(x_i^{1m} - x_i^j)|} - \arctan\frac{(x_i^{1m} - x_i^j)t_i^m}{|n_i^m(x_i^{1m} - x_i^j)|}\right) \end{array} \right\} \\ &\quad + \frac{n_i^1(x_i^{11} - x_i^j)}{2\pi l_1} \left\{ \begin{array}{c} \frac{1}{2}\ln\left(1 + 2\frac{l_1(x_i^{11} - x_i^j)t_i^1}{|\mathbf{x}^{11} - \mathbf{x}^j|^2} + \frac{l_1^2}{|\mathbf{x}^{11} - \mathbf{x}^j|^2}\right) \\ -\frac{l_1 + (x_i^{11} - x_i^j)t_i^1}{|n_i^1(x_i^{11} - x_i^j)|}\left(\begin{array}{c}\arctan\frac{l_1 + (x_i^{11} - x_i^j)t_i^1}{|n_i^1(x_i^{11} - x_i^j)|} \\ - \arctan\frac{(x_i^{11} - x_i^j)t_i^1}{|n_i^1(x_i^{11} - x_i^j)|}\end{array}\right) \end{array} \right\} \end{aligned} \tag{4.70}$$

Related to the heat fluxes q_n^{1e} and q_n^{2e} respectively, at x_i^j one obtains from (3.148) the both contributions from the adjacent elements Γ^e and Γ^{e+1}

$$H_{je}^2 = -\frac{l_e}{4\pi\lambda_0}\int_{\eta=0}^{1}\eta\left\{\ln\left(\eta^2 + \frac{2}{l_e}\mathbf{t}^e \cdot \mathbf{r}^{1ej}\eta + \frac{\mid \mathbf{r}^{1ej} \mid^2}{l_e^2}\right) + \ln\left(\frac{l_e^2}{l_0^2}\right)\right\}d\eta \tag{4.71}$$

$$H_{je+1}^1 = -\frac{l_{e+1}}{4\pi\lambda_0}\int_{\eta=0}^{1}(1-\eta)\left\{\ln\left(\eta^2 + \frac{2}{l_{e+1}}\mathbf{t}^{e+1} \cdot \mathbf{r}^{1e+1j}\eta + \frac{\mid \mathbf{r}^{1e+1j} \mid^2}{l_{e+1}^2}\right) + \ln\left(\frac{l_{e+1}^2}{l_0^2}\right)\right\}d\eta \tag{4.72}$$

where these two contributions are only added to $H_{jk} = H^2_{je} + H^1_{je+1}$, $k = e + 1$, if q^k_n is continous at this point. This gives explicitly:

for $x^j_i = x^{1e+1}_i = x^{2e}_i$
i.e., $x^{1e}_i - x^j_i = x^{1e}_i - x^{2e}_i = -l_e t^e_i \rightarrow \mathbf{t}^e \cdot \mathbf{r}^{1ej} = t^e_i(-l_e t^e_i) = -l_e$ and $\mid \mathbf{r}^{1ej} \mid = l_e$
and $x^{1e+1}_i - x^j_i = x^{1e+1}_i - x^{2e}_i = 0 \rightarrow \mathbf{t}^{e+1} \cdot \mathbf{r}^{1e+1j} = 0$ and $\mid \mathbf{r}^{1e+1j} \mid = 0$

$$H^2_{je} = -\frac{l_e}{4\pi\lambda_0} \lim_{\varepsilon\to 0} \int_{\eta=0}^{1-\varepsilon} \eta \left\{ \ln\left(\eta^2 - 2\eta + 1\right) + \ln\left(\frac{l_e^2}{l_0^2}\right) \right\} d\eta = \frac{l_e}{8\pi\lambda_0} \left\{ 3 - \ln\left(\frac{l_e^2}{l_0^2}\right) \right\} \quad (4.73)$$

$$H^1_{je+1} = -\frac{l_{e+1}}{4\pi\lambda_0} \lim_{\varepsilon\to 0} \int_{\eta=\varepsilon}^{1} (1-\eta) \left\{ \ln\left(\eta^2\right) + \ln\left(\frac{l_{e+1}^2}{l_0^2}\right) \right\} d\eta = \frac{l_{e+1}}{8\pi\lambda_0} \left\{ 3 - \ln\left(\frac{l_{e+1}^2}{l_0^2}\right) \right\} \quad (4.74)$$

for $x^j_i = x^{1e}_i$
i.e., $x^{1e}_i - x^j_i = 0$

$$H^2_{je} = -\frac{l_e}{4\pi\lambda_0} \lim_{\varepsilon\to 0} \int_{\eta=\varepsilon}^{1} \eta \left\{ \ln\left(\eta^2\right) + \ln\left(\frac{l_e^2}{l_0^2}\right) \right\} d\eta = \frac{l_e}{8\pi\lambda_0} \left\{ 1 - \ln\left(\frac{l_e^2}{l_0^2}\right) \right\} \quad (4.75)$$

for $x^j_i = x^{2e+1}_i$
i.e., $x^{1e+1}_i - x^j_i = x^{1e+1}_i - x^{2e+1}_i = -l_{e+1} t^{e+1}_i \rightarrow \mathbf{t}^{e+1} \cdot \mathbf{r}^{1e+1j} = t^{e+1}_i(-l_{e+1} t^{e+1}_i) = -l_{e+1}$ and $\mid \mathbf{r}^{1e+1j} \mid = l_{e+1}$

$$H^1_{je+1} = -\frac{l_{e+1}}{4\pi\lambda_0} \lim_{\varepsilon\to 0} \int_{\eta=0}^{1-\varepsilon} (1-\eta) \left\{ \ln\left(\eta^2 - 2\eta + 1\right) + \ln\left(\frac{l_{e+1}^2}{l_0^2}\right) \right\} d\eta = \frac{l_{e+1}}{8\pi\lambda_0} \left\{ 1 - \ln\left(\frac{l_{e+1}^2}{l_0^2}\right) \right\} \quad (4.76)$$

for $x^j_i \neq x^{1e}_i$ **and** $x^j_i \neq x^{2e}_i = x^{1e+1}_i$ **and** $n^e_l(x^{1e}_l - x^j_l) = \mathbf{n}^e \cdot \mathbf{r}^{1ej} \neq 0:$

$$H^2_{je} = \frac{-l_e}{8\pi\lambda_0} \left\{ \begin{array}{c} \left(\begin{array}{c} \frac{|\mathbf{n}^e\cdot\mathbf{r}^{1ej}|^2}{l_e^2} \\ -\frac{|\mathbf{t}^e\cdot\mathbf{r}^{1ej}|^2}{l_e^2} \end{array} \right) \ln\left(1 + 2\frac{l_e \mathbf{t}^e\cdot\mathbf{r}^{1ej}}{|\mathbf{r}^{1ej}|^2} + \frac{l_e^2}{|\mathbf{r}^{1ej}|^2}\right) + \ln\left(\frac{l_e^2}{l_0^2}\right) \\ + \ln\left(1 + \frac{2}{l_e}\mathbf{t}^e\cdot\mathbf{r}^{1ej} + \frac{|\mathbf{r}^{1ej}|^2}{l_e^2}\right) + \frac{2}{l_e}\mathbf{t}^e\cdot\mathbf{r}^{1ej} - 1 \\ -\frac{4}{l_e^2}\mathbf{t}^e\cdot\mathbf{r}^{1ej} \mid \mathbf{n}^e\cdot\mathbf{r}^{1ej} \mid \left(\begin{array}{c} \arctan\frac{l_e + \mathbf{t}^e\cdot\mathbf{r}^{1ej}}{|\mathbf{n}^e\cdot\mathbf{r}^{1ej}|} \\ -\arctan\frac{\mathbf{t}^e\cdot\mathbf{r}^{1ej}}{|\mathbf{n}^e\cdot\mathbf{r}^{1ej}|} \end{array} \right) \end{array} \right\} \quad (4.77)$$

for $x^j_i \neq x^{1e+1}_i$ **and** $n^{e+1}_l(x^{1e+1}_l - x^j_l) = \mathbf{n}^{e+1} \cdot \mathbf{r}^{1e+1j} \neq 0$ with $e + 1 = k$

$$H^1_{jk} = \frac{-l_k}{8\pi\lambda_0} \left\{ \begin{array}{c} \left(\begin{array}{c} \frac{2}{l_k}\mathbf{t}^k\cdot\mathbf{r}^{1kj} \\ -\frac{|\mathbf{n}^k\cdot\mathbf{r}^{1kj}|^2}{l_k^2} \\ +\frac{|\mathbf{t}^k\cdot\mathbf{r}^{1kj}|^2}{l_k^2} \end{array} \right) \ln\left(1 + 2\frac{l_k \mathbf{t}^k\cdot\mathbf{r}^{1kj}}{|\mathbf{r}^{1kj}|^2} + \frac{l_k^2}{|\mathbf{r}^{1kj}|^2}\right) + \ln\left(\frac{l_k^2}{l_0^2}\right) \\ + \ln\left(1 + \frac{2}{l_k}\mathbf{t}^k\cdot\mathbf{r}^{1kj} + \frac{|\mathbf{r}^{1kj}|^2}{l_k^2}\right) - \frac{2}{l_k}\mathbf{t}^k\cdot\mathbf{r}^{1kj} - 3 \\ +\frac{4}{l_k}\left(\begin{array}{c} 1+ \\ \frac{1}{l_k}\mathbf{t}^k\cdot\mathbf{r}^{1kj} \end{array} \right) \mid \mathbf{n}^k\cdot\mathbf{r}^{1kj} \mid \left(\begin{array}{c} \arctan\frac{l_k + \mathbf{t}^k\cdot\mathbf{r}^{1kj}}{|\mathbf{n}^k\cdot\mathbf{r}^{1kj}|} \\ -\arctan\frac{\mathbf{t}^k\cdot\mathbf{r}^{1kj}}{|\mathbf{n}^k\cdot\mathbf{r}^{1kj}|} \end{array} \right) \end{array} \right\} \quad (4.78)$$

Remark:

a) Integration by parts yields the following result which was important for the integration of G_{je} (for $4b - a^2 > 0$):

$$\int \ln(\eta^2 + a\eta + b)\, d\eta = \begin{array}{c}(\eta + \frac{a}{2}) \ln(\eta^2 + a\eta + b) - 2\eta \\ +\sqrt{4b - a^2} \arctan \frac{2\eta + a}{\sqrt{4b-a^2}}\end{array}$$

$$\int \eta \ln(\eta^2 + a\eta + b)\, d\eta = \frac{1}{2} \left[\begin{array}{c}(\eta^2 + b - \frac{a^2}{2}) \ln(\eta^2 + a\eta + b) + a\eta - \eta^2 \\ -a\sqrt{4b - a^2} \arctan \frac{2\eta + a}{\sqrt{4b-a^2}}\end{array} \right]$$

b) In evaluating the integrals, one obtains the following difference of two ln-terms which can be assembled as follows:

$$\begin{aligned} & \ln\left(1 + \frac{2}{l_e}\mathbf{t}^e \cdot \mathbf{r}^{1ej} + \frac{|\ \mathbf{r}^{1ej}\ |^2}{l_e^2}\right) - \ln\left(\frac{|\ \mathbf{r}^{1ej}\ |^2}{l_e^2}\right) \\ = \ & \ln\left(1 + 2\frac{l_e \mathbf{t}^e \cdot \mathbf{r}^{1ej}}{|\ \mathbf{r}^{1ej}\ |^2} + \frac{l_e^2}{|\ \mathbf{r}^{1ej}\ |^2}\right) \end{aligned}$$

c) It is well known that the scalar product of a directional unit vector with an arbitrary vector gives the vector's component in the direction of the unit vector. Hence, for the unit vectors $\mathbf{t}^e$ and $\mathbf{n}^e$, which are orthogonal to each other, holds (see Fig.)

$$|\ \mathbf{r}^{1ej}\ |^2 = |\ \mathbf{t}^e \cdot \mathbf{r}^{1ej}\ |^2 + |\ \mathbf{n}^e \cdot \mathbf{r}^{1ej}\ |^2$$

4.3.2.2 Exercise 15: Stationary heat conduction in a rectangular domain-linear shape functions

In a rectangular domain $\Omega = \{(x_1, x_2) \mid 0 \le x_1 \le l_1 = 1m,\ 0 \le x_2 \le l_2 = 2m\}$, the temperature is prescribed at two opposite sides $x_2 = 0$ and $x_2 = l_2$ to be $\Theta(x_1, 0) = 0$ and $\Theta(x_1, l_2) = \overline{\Theta} > 0$ while at the other two sides $x_1 = 0$ and $x_1 = l_1$ the heat flux is stopped, i.e., $q_n = 0$. For this problem (see [2]), the exact temperature distribution is known to be $\Theta(x_1, x_2) = \overline{\Theta}\ x_2/l_2$ from which one can determine via (3.146) the heat flux across the boundary to be as $q_n(x_1, x_2) = \lambda_0 n_i(x_1, x_2) \partial\Theta(x_1, x_2)/\partial x_i = \lambda_0 n_2(x_1, x_2)\overline{\Theta}\ /l_2$.

Solve this problem applying the above given linear boundary element approximation by taken each of the four sides as a boundary element and compare with the known exact solution

5 Appendices

5.1 A: Exercise Solutions

5.1.0.3 Exercise 1: Nabla Vector

$$\begin{aligned}
\mathrm{grad}U &= \frac{\partial U}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial U}{\partial x_2}\hat{\mathbf{e}}_2 \\
&= \left(\frac{\partial U}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial U}{\partial \varphi}\frac{\partial \varphi}{\partial x_1}\right)(\hat{\mathbf{e}}_r\cos\varphi - \hat{\mathbf{e}}_\varphi\sin\varphi) \\
&\quad + \left(\frac{\partial U}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial U}{\partial \varphi}\frac{\partial \varphi}{\partial x_2}\right)(\hat{\mathbf{e}}_r\sin\varphi + \hat{\mathbf{e}}_\varphi\cos\varphi)
\end{aligned}$$

Since

$$\begin{aligned}
r &= \sqrt{x_1^2 + x_1^2} \rightarrow \frac{\partial r}{\partial x_1} = \frac{x_1}{r} = \cos\varphi,\ \frac{\partial r}{\partial x_2} = \frac{x_2}{r} = \sin\varphi \\
\varphi &= \arctan(\frac{x_2}{x_1}) \rightarrow \frac{\partial \varphi}{\partial x_1} = \frac{-x_2}{r^2} = -\frac{\sin\varphi}{r},\ \frac{\partial \varphi}{\partial x_2} = \frac{x_1}{r^2} = \frac{\cos\varphi}{r}
\end{aligned}$$

the above expression for $\mathrm{grad}U$ may be expressed as

$$\begin{aligned}
\mathrm{grad}U &= \left(\frac{\partial U}{\partial r}\cos\varphi + \frac{\partial U}{\partial \varphi}(-\frac{\sin\varphi}{r})\right)(\hat{\mathbf{e}}_r\cos\varphi - \hat{\mathbf{e}}_\varphi\sin\varphi) \\
&\quad + \left(\frac{\partial U}{\partial r}\sin\varphi + \frac{\partial U}{\partial \varphi}\frac{\cos\varphi}{r}\right)(\hat{\mathbf{e}}_r\sin\varphi + \hat{\mathbf{e}}_\varphi\cos\varphi) \\
&= \frac{\partial U}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial U}{\partial \varphi}\hat{\mathbf{e}}_\varphi \\
&= (\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\varphi\frac{1}{r}\frac{\partial}{\partial \varphi})U
\end{aligned}$$

5.1.0.4 Exercise 2: Laplace operator

$$\Delta U = (\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\varphi\frac{1}{r}\frac{\partial}{\partial \varphi}) \cdot (\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\varphi\frac{1}{r}\frac{\partial}{\partial \varphi})U$$

The scalar product is only 'working' between unit vectors, i.e., $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\varphi \cdot \hat{\mathbf{e}}_\varphi = 1$ and $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_\varphi = 0$, but the differentiations have to be performed following the chain rule, i.e.,

$$\begin{aligned}
\frac{\partial}{\partial r}(\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\varphi\frac{1}{r}\frac{\partial}{\partial \varphi}) &= \frac{\partial \hat{\mathbf{e}}_r}{\partial r}\frac{\partial}{\partial r} + \hat{\mathbf{e}}_r\frac{\partial^2}{\partial r^2} + \frac{\partial \hat{\mathbf{e}}_\varphi}{\partial r}\frac{1}{r}\frac{\partial}{\partial \varphi} + \hat{\mathbf{e}}_\varphi(-\frac{1}{r^2}\frac{\partial}{\partial \varphi} + \frac{1}{r}\frac{\partial^2}{\partial \varphi \partial r}) \\
\frac{\partial}{\partial \varphi}(\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\varphi\frac{1}{r}\frac{\partial}{\partial \varphi}) &= \frac{\partial \hat{\mathbf{e}}_r}{\partial \varphi}\frac{\partial}{\partial r} + \hat{\mathbf{e}}_r\frac{\partial^2}{\partial r \partial \varphi} + \frac{\partial \hat{\mathbf{e}}_\varphi}{\partial \varphi}\frac{1}{r}\frac{\partial}{\partial \varphi} + \hat{\mathbf{e}}_\varphi\frac{1}{r}\frac{\partial^2}{\partial \varphi^2}
\end{aligned}$$

For the differentiations of the unit vectors of the Polar coordinate system holds

$$\frac{\partial \hat{\mathbf{e}}_r}{\partial r} = \frac{\partial \hat{\mathbf{e}}_\varphi}{\partial r} = 0,\ \frac{\partial \hat{\mathbf{e}}_r}{\partial \varphi} = \hat{\mathbf{e}}_\varphi,\ \frac{\partial \hat{\mathbf{e}}_\varphi}{\partial \varphi} = -\hat{\mathbf{e}}_r$$

which reduces the above results to

$$\begin{aligned}
\frac{\partial}{\partial r}(\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\varphi\frac{1}{r}\frac{\partial}{\partial \varphi}) &= \hat{\mathbf{e}}_r\frac{\partial^2}{\partial r^2} + \hat{\mathbf{e}}_\varphi(-\frac{1}{r^2}\frac{\partial}{\partial \varphi} + \frac{1}{r}\frac{\partial^2}{\partial \varphi \partial r}) \\
\frac{\partial}{\partial \varphi}(\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\varphi\frac{1}{r}\frac{\partial}{\partial \varphi}) &= \hat{\mathbf{e}}_\varphi\frac{\partial}{\partial r} + \hat{\mathbf{e}}_r\frac{\partial^2}{\partial r \partial \varphi} - \hat{\mathbf{e}}_r\frac{1}{r}\frac{\partial}{\partial \varphi} + \hat{\mathbf{e}}_\varphi\frac{1}{r}\frac{\partial^2}{\partial \varphi^2}
\end{aligned}$$

Now, performing the scalar product of the first line result with $\hat{\mathbf{e}}_r$ and of the second one with $\hat{\mathbf{e}}_\varphi\frac{1}{r}$ gives finally

$$\Delta U = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2})U$$

5.1.0.5 Exercise 3: Integration by parts

a) in R^1 on the intervall $a < x < b$:

$$\begin{aligned}
\int\limits_a^b x^n \ln(x)dx &= \left[\frac{x^{n+1}}{n+1}\ln(x)\right]_{x=a}^{x=b} - \int\limits_a^b \frac{x^{n+1}}{n+1}\frac{1}{x}dx \\
&= \left[\frac{x^{n+1}}{n+1}\ln(x) - \frac{x^{n+1}}{(n+1)^2}\right]_{x=a}^{x=b}
\end{aligned}$$

b) in R^2 on the circular domain $\Omega = \left\{(x_1, x_2) \mid r = \sqrt{x_1^2 + x_2^2} \leq R\right\}$:

$$\begin{aligned}
\int\limits_\Omega r^n \ln(r)d\Omega &= \int\limits_0^{2\pi}\int\limits_0^R r^n \ln(r)rdrd\varphi = 2\pi\int\limits_0^R r^{n+1}\ln(r)dr \\
&= 2\pi\left[\frac{r^{n+2}}{n+2}\ln(r)\right]_0^R - 2\pi\int\limits_0^R \frac{r^{n+2}}{n+2}\frac{1}{r}dr \\
&= 2\pi\left[\frac{r^{n+2}}{n+2}\ln(r) - \frac{r^{n+2}}{(n+2)^2}\right]_0^R \\
&= \frac{2\pi}{n+2}R^{n+2}\left(\ln(R) - \frac{1}{n+2}\right)
\end{aligned}$$

5.1.0.6 Exercise 4: Reversed order integrations

a)

$$\int_a^x \left[\int_a^s \bar{x}^3 s\, d\bar{x}\right] ds = \int_a^x \left[\frac{\bar{x}^4}{4}s\right]_{\bar{x}=a}^{\bar{x}=s} ds = \int_a^x \left[\frac{s^5}{4} - \frac{a^4}{4}s\right] ds = \left[\frac{s^6}{24} - \frac{a^4}{8}s^2\right]_{s=a}^{s=x}$$
$$= \frac{x^6}{24} - \frac{1}{8}a^4x^2 - \frac{a^6}{24} + \frac{a^6}{8} = \frac{1}{24}(x^6 - 3a^4x^2 + 2a^6)$$

$$\int_a^x \left[\int_{\bar{x}}^x \bar{x}^3 s\, ds\right] d\bar{x} = \int_a^x \left[\bar{x}^3\frac{s^2}{2}\right]_{s=\bar{x}}^{s=x} d\bar{x} = \int_a^x \left[\bar{x}^3\frac{x^2}{2} - \frac{\bar{x}^5}{2}\right] d\bar{x} = \left[\frac{1}{8}\bar{x}^4x^2 - \frac{\bar{x}^6}{12}\right]_{\bar{x}=a}^{\bar{x}=x}$$
$$= \frac{x^6}{8} - \frac{x^6}{12} - \frac{1}{8}a^4x^2 + \frac{a^6}{12} = \frac{1}{24}(x^6 - 3a^4x^2 + 2a^6)$$

b)

$$\int_a^x \left[\int_s^b G(\bar{x}, s)d\bar{x}\right] ds = \int_a^x \left[\int_s^x G(\bar{x}, s)d\bar{x}\right] ds + \int_a^x \left[\int_x^b G(\bar{x}, s)d\bar{x}\right] ds$$
$$= \int_a^x \left[\int_a^{\bar{x}} G(\bar{x}, s)ds\right] d\bar{x} + \int_x^b \left[\int_a^x G(\bar{x}, s)ds\right] d\bar{x} \qquad (5.1)$$

$$\int_a^x \left[\int_s^b \bar{x}^3 s\, d\bar{x}\right] ds = \int_a^x \left[\frac{\bar{x}^4}{4}s\right]_{\bar{x}=s}^{\bar{x}=b} ds = \int_a^x \left[\frac{b^4 s}{4} - \frac{s^5}{4}\right] ds = \left[\frac{b^4s^2}{8} - \frac{s^6}{24}\right]_{s=a}^{s=x}$$
$$= \frac{b^4x^2}{8} - \frac{x^6}{24} - \frac{b^4a^2}{8} + \frac{a^6}{24} = \frac{1}{24}(3b^4x^2 - x^6 - 3b^4a^2 + a^6)$$

$$\int_a^x \left[\int_a^{\bar{x}} \bar{x}^3 s\, ds\right] d\bar{x} + \int_x^b \left[\int_a^x \bar{x}^3 s\, ds\right] d\bar{x} = \int_a^x \left[\bar{x}^3\frac{s^2}{2}\right]_{s=a}^{s=\bar{x}} d\bar{x} + \int_x^b \left[\bar{x}^3\frac{s^2}{2}\right]_{s=a}^{s=x} d\bar{x}$$
$$= \int_a^x \left[\frac{\bar{x}^5}{2} - \bar{x}^3\frac{a^2}{2}\right] d\bar{x} + \int_x^b \left[\bar{x}^3\frac{x^2}{2} - \bar{x}^3\frac{a^2}{2}\right] d\bar{x}$$
$$= \left[\frac{\bar{x}^6}{12} - \frac{\bar{x}^4}{4}\frac{a^2}{2}\right]_{\bar{x}=a}^{\bar{x}=x} + \left[\frac{\bar{x}^4}{4}(\frac{x^2}{2} - \frac{a^2}{2})\right]_{\bar{x}=x}^{\bar{x}=b}$$
$$= [\frac{x^6}{12} - \frac{x^4a^2}{8} - \frac{a^6}{12} + \frac{a^6}{8}] + [\frac{b^4}{4}(\frac{x^2}{2} - \frac{a^2}{2}) - \frac{x^4}{4}(\frac{x^2}{2} - \frac{a^2}{2})]$$
$$= \frac{1}{24}(3b^4x^2 - x^6 - 3b^4a^2 + a^6)$$

5.1.0.7 Exercise 5: Integral equation by straightforward integrations

One integration of both sides of the differential equation (3.8) gives

$$\frac{dy(x)}{dx} = \left[\int f(\bar{x})d\bar{x}\right]_{\bar{x}=x} + c_0$$

and satisfying the boundary condition $y'(b) = y'_1$

$$\frac{dy(x)}{dx} = y'_1 - \int_x^b f(\bar{x})d\bar{x}$$

Then, a second produces

$$y(x) = y'_1 x - \left[\int \left(\int_s^b f(\bar{x})d\bar{x} \right) ds \right]_{s=x} + c_1$$

and finally

$$y(x) = y_0 + (x-a)y'_1 - \int_a^x \left[\int_s^b f(\bar{x})d\bar{x} \right] ds$$

when the further constant of integration c_1 having been taken so that $y(a) = y_0$. Applying the formula (5.1), derived in Exercise 2, to the above double integral gives

$$\begin{aligned} \int_a^x \left[\int_s^b f(\bar{x})d\bar{x} \right] ds &= \int_a^x \left[\int_a^{\bar{x}} f(\bar{x})ds \right] d\bar{x} + \int_x^b \left[\int_a^x f(\bar{x})ds \right] d\bar{x} \\ &= \int_a^x (\bar{x}-a)f(\bar{x})d\bar{x} + \int_x^b (x-a)f(\bar{x})d\bar{x} \end{aligned}$$

and the final simplified expression for the solution of the boundary value problem:

$$y(x) = y_0 + (x-a)y'_1 - \int_a^x (\bar{x}-a)f(\bar{x})d\bar{x} - (x-a)\int_x^b f(\bar{x})d\bar{x}. \tag{5.2}$$

5.1.0.8 Exercise 6: Beam deflection under prescribed moments

When in (3.28) the prescribed boundary conditions are taken into account and the system is reordered, one obtains

$$\begin{bmatrix} 0 & -1 \\ -l & 1 \end{bmatrix} \begin{bmatrix} w'(a) \\ w(b) \end{bmatrix} = -\frac{1}{EI}\int_a^b \begin{bmatrix} (x-a)M(x) \\ (b-x)M(x) \end{bmatrix} dx - \begin{bmatrix} 1 & l \\ -1 & 0 \end{bmatrix} \begin{bmatrix} w_0 \\ w'_1 \end{bmatrix}$$

which has the solutions $(b - a = l)$

$$\begin{aligned} w(b) &= \frac{1}{EI}\int_a^b (x-a)M(x)dx + w_0 + lw'_1 \\ w'(a) &= \frac{1}{l}\left(\frac{1}{EI}\int_a^b (b-x)M(x)dx - w_0 + w(b) \right) \\ &= \frac{1}{l}\left(\frac{1}{EI}\int_a^b (b-x)M(x)dx - w_0 + \frac{1}{EI}\int_a^b (x-a)M(x)dx + w_0 + lw'_1 \right) \\ &= \frac{1}{l}\left(\frac{1}{EI}\int_a^b (b-a)M(x)dx + lw'_1 \right) \\ &= \frac{1}{EI}\int_a^b M(x)dx + w'_1 \end{aligned}$$

With these boundary reactions and the prescribed boundary values, the solution (3.27) becomes

$$\begin{aligned}
w(\xi) &= \frac{1}{2}w(a) + \frac{\xi - a}{2}w^{'}(a) + \frac{1}{2}w(b) - \frac{(b-\xi)}{2}w^{'}(b) - \int_a^b \frac{M(x)}{EI}\frac{1}{2} \mid x - \xi \mid dx \\
&= \frac{1}{2}w_0 + \frac{\xi - a}{2}\left(\frac{1}{EI}\int_a^b M(x)dx + w_1^{'}\right) + \frac{1}{2}\left(\frac{1}{EI}\int_a^b (x-a)M(x)dx + w_0 + lw_1^{'}\right) \\
&\quad -\frac{(b-\xi)}{2}w_1^{'} - \int_a^b \frac{M(x)}{EI}\frac{1}{2}\left|x-\xi\right| dx \\
&= w_0 + (\xi - a)w_1^{'} + \frac{1}{2}\frac{1}{EI}\int_a^b \left(\xi - 2a + x - \left|x - \xi\right|\right) M(x)dx \\
&= w_0 + (\xi - a)w_1^{'} + \frac{1}{2}\frac{1}{EI}\int_a^\xi \left(2x - 2a\right) M(x)dx + \frac{1}{2}\frac{1}{EI}\int_\xi^b \left(2\xi - 2a\right) M(x)dx \\
&= w_0 + (\xi - a)w_1^{'} + \frac{1}{EI}\int_a^\xi \left(x - a\right) M(x)dx + \frac{1}{EI}(\xi - a)\int_\xi^b M(x)dx
\end{aligned}$$

This is exactly the solution (5.2) when one recognizes that ξ and x are there x and $\bar{x}$, respectively, and $f = -M/EI$.

5.1.0.9 Exercise 7: Bending moment of an elastic beams under transversal loading

By formally setting M, Q, and q instead of w, $w^{'}$, and M/EI, respectively, the solution is directly obtained from (3.27) as

$$M(\xi) = \frac{1}{2}M(a) + \frac{\xi - a}{2}Q(a) + \frac{1}{2}M(b) - \frac{(b-\xi)}{2}Q(b) - \int_a^b q(x)\frac{1}{2} \mid x - \xi \mid dx \qquad (5.3)$$

while the algebraic system (3.28) to determine the unknown boundary reactions becomes $(b - a = l)$:

$$\begin{bmatrix} 1 & 0 & -1 & l \\ -1 & -l & 1 & 0 \end{bmatrix} \begin{bmatrix} M(a) \\ Q(a) \\ M(b) \\ Q(b) \end{bmatrix} = -\int_a^b \begin{bmatrix} (x-a)q(x) \\ (b-x)q(x) \end{bmatrix} dx \qquad (5.4)$$

5.1.0.10 Exercise 8: Axial displacement of elastic bars

By formally setting u, N/EA, and p/EA instead of w, $w^{'}$, and M/EI, respectively, the solution is directly obtained from (3.24)

$$u(\xi) = -\left[\frac{du(x)}{dx}u^*(x,\xi) - u(x)\frac{\partial u^*(x,\xi)}{\partial x}\right]_a^b - \int_a^b \frac{p(x)}{EA}u^*(x,\xi)dx \qquad (5.5)$$

and with $N(x) = EAu'(x)$

$$u(\xi) = -\left[\frac{N(x)}{EA}u^*(x,\xi) - u(x)\frac{N^*(x,\xi)}{EA}\right]_a^b - \int_a^b \frac{p(x)}{EA}u^*(x,\xi)dx \tag{5.6}$$

or in detail with (3.27) as

$$u(\xi) = \frac{1}{2}u(a) + \frac{\xi - a}{2}\frac{N(a)}{EA} + \frac{1}{2}u(b) - \frac{(b-\xi)}{2}\frac{N(b)}{EA} - \int_a^b \frac{p(x)}{EA}\frac{1}{2} \mid x - \xi \mid dx \tag{5.7}$$

Then, the algebraic system (3.28) to determine the unknown boundary reactions becomes $(b - a = l)$:

$$\begin{bmatrix} 1 & 0 & -1 & l \\ -1 & -l & 1 & 0 \end{bmatrix} \begin{bmatrix} u(a) \\ \frac{N(a)}{EA} \\ u(b) \\ \frac{N(b)}{EA} \end{bmatrix} = -\frac{1}{EA}\int_a^b \begin{bmatrix} (x-a)p(x) \\ (b-x)p(x) \end{bmatrix} dx \tag{5.8}$$

5.1.0.11 Exercise 9: Bar stretching under axial loadings

The system (5.8) becomes for this case

$$\begin{bmatrix} 0 & -1 \\ -l & 1 \end{bmatrix} \begin{bmatrix} \frac{N(0)}{EA} \\ u(l) \end{bmatrix} = -\frac{1}{EA}\int_0^l \begin{bmatrix} x\,p_0\frac{x}{l} \\ (l-x)p_0\frac{x}{l} \end{bmatrix} dx = -\frac{p_0}{EA}\begin{bmatrix} \frac{l^2}{3} \\ \frac{l^2}{6} \end{bmatrix}$$

and its solutions are

$$\begin{aligned} u(l) &= \frac{p_0}{EA}\frac{l^2}{3} \\ N(0) &= p_0\frac{l}{6} + \frac{EA}{l}u(l) = p_0\left(\frac{l}{6} + \frac{l}{3}\right) = p_0\frac{l}{2} \end{aligned}$$

Inserting the prescribed boundary conditions and these boundary reaction in the solution (5.7) gives

$$\begin{aligned} u(\xi) &= \frac{\xi}{2}\frac{N(0)}{EA} + \frac{1}{2}u(l) - \int_0^l \frac{p_0}{EA}\frac{x}{l}\frac{1}{2} \mid x - \xi \mid dx \\ &= \frac{\xi}{2}\frac{p_0 l}{2EA} + \frac{1}{2}\frac{p_0}{EA}\frac{l^2}{3} - \int_0^\xi \frac{p_0}{EA}\frac{x}{l}\frac{1}{2}(\xi - x)dx - \int_\xi^l \frac{p_0}{EA}\frac{x}{l}\frac{1}{2}(x-\xi)dx \\ &= \frac{p_0}{EA}\left(\frac{\xi l}{4} + \frac{l^2}{6} - \frac{\xi^3}{12l} - \frac{1}{12l}(2l^3 - 3\xi l^2 + \xi^3)\right) \\ &= \frac{p_0}{6EA}\left(3\xi l - \frac{\xi^3}{l}\right) \end{aligned}$$

5.1.0.12 Exercise 10: Torsional twist of an elastic bar

The adequate fundamental solution of this Helmholtz type equation is (3.39)

$$\vartheta^*(x,\xi) = \vartheta^*(r) = \frac{-1}{2h}\mathrm{e}^{-hr}$$

which has the first derivative

$$\frac{\partial \vartheta^*(x,\xi)}{\partial x} = \frac{1}{2}e^{-hr}\frac{\partial r}{\partial x} = \frac{1}{2}e^{-hr}\mathrm{sign}(x-\xi)$$

Hence, substituting the corresponding fundamental solution terms and introducing the actual state variables $\vartheta(x)$ and $\vartheta'(x)$ in the integral form of the solution (3.53) yields

$$\vartheta(\xi) = \frac{1}{2h}\mathrm{e}^{-h(l-\xi)}\vartheta'(l) + \frac{1}{2}\mathrm{e}^{-h(l-\xi)}\vartheta(l) - \frac{1}{2h}\mathrm{e}^{-h\xi}\vartheta'(0) + \frac{1}{2}\mathrm{e}^{-h\xi}\vartheta(0) + \int_0^l \frac{M_T(x)}{EC_T 2h}\mathrm{e}^{-h|x-\xi|}dx$$

and, correspondingly, one obtains from (3.55)

$$\begin{bmatrix} 1 & 1 & -\mathrm{e}^{-hl} & -\mathrm{e}^{-hl} \\ -\mathrm{e}^{-hl} & \mathrm{e}^{-hl} & 1 & -1 \end{bmatrix} \begin{bmatrix} \vartheta(0) \\ \frac{1}{h}\vartheta'(0) \\ \vartheta(l) \\ \frac{1}{h}\vartheta'(l) \end{bmatrix} = \frac{1}{EC_T h}\int_0^l \begin{bmatrix} M_T(x)\mathrm{e}^{-hx} \\ M_T(x)\mathrm{e}^{-h|x-\xi|} \end{bmatrix} dx$$

5.1.0.13 Exercise 11: Green' functions for beam problems

a) When the beam has a clamped support at both endings, the boundary conditions for the Green's function of this boundary value problem are

$$\begin{aligned} G^{cc}(0,\xi) &= 0 \quad \text{and} \quad G^{cc}(l,\xi) = 0 \\ \left.\frac{\partial G^{cc}(x,\xi)}{\partial x}\right|_{x=0} &= 0 \quad \text{and} \quad \left.\frac{\partial G^{cc}(x,\xi)}{\partial x}\right|_{x=l} = 0 \end{aligned}$$

With the polynomial 'ansatz'

$$G^{cc}(x,\xi) = \frac{1}{12EI}\left\{r^3 + c_1x^3 + c_2x^2 + c_3x + c_4\right\}$$

one easily finds the following conditions for the four free constants c_1, c_2, c_3, and c_4 ($r = |x-\xi|$, $0 \le \xi \le l$)

$$\begin{aligned} G^{cc}(0,\xi) &= 0 = \xi^3 + c_4 \\ G^{cc}(l,\xi) &= 0 = (l-\xi)^3 + c_1l^3 + c_2l^2 + c_3l + c_4 \\ \left.\frac{\partial G^{cc}(x,\xi)}{\partial x}\right|_{x=0} &= 0 = 3r^2\mathrm{sign}(x-\xi)\big|_{x=0} + c_3 \\ &= -3\xi^2 + c_3 \\ \left.\frac{\partial G^{cc}(x,\xi)}{\partial x}\right|_{x=l} &= 0 = 3r^2\mathrm{sign}(x-\xi)\big|_{x=l} + 3c_1l^2 + 2c_2l + c_3 \\ &= 3(l-\xi)^2 + 3c_1l^2 + 2c_2l + c_3 \end{aligned}$$

Solving these four equations gives

$$c_1 = -1 + 6\left(\frac{\xi}{l}\right)^2 - 4\left(\frac{\xi}{l}\right)^3, \; c_2 = 3\xi\left[1 - 4\left(\frac{\xi}{l}\right) + 2\left(\frac{\xi}{l}\right)^2\right] \; c_3 = 3\xi^2 \; c_4 = -\xi^3$$

and, finally, the following Green's function

$$\begin{aligned} G^{cc}(x,\xi) &= \frac{1}{12EI}\left\{\begin{array}{c} r^3 + \left[-1 + 6\left(\frac{\xi}{l}\right)^2 - 4\left(\frac{\xi}{l}\right)^3\right]x^3 \\ + \left[1 - 4\left(\frac{\xi}{l}\right) + 2\left(\frac{\xi}{l}\right)^2\right]3\xi x^2 + 3\xi^2 x - \xi^3 \end{array}\right\} \\ &= \frac{1}{12EI}\left\{\begin{array}{c} r^3 - x^3 + 3\xi x^2 + 3\xi^2 x - \xi^3 + [6l\xi^2 - 4\xi^3]\left(\frac{x}{l}\right)^3 \\ + [2\xi^2 - 4l\xi]\,3\xi\left(\frac{x}{l}\right)^2 \end{array}\right\} \end{aligned}$$

b) For the Green's function of a beam with a clamped support at $x = 0$ while the other ending is free, one starts again with the polynomial 'ansatz'

$$G^{cf}(x,\xi) = \frac{1}{12EI}\left\{r^3 + c_1 x^3 + c_2 x^2 + c_3 x + c_4\right\}$$

There, the following conditions hold

$$\begin{aligned} G^{cf}(0,\xi) &= 0 = \xi^3 + c_4 \\ \left.\frac{\partial G^{cf}(x,\xi)}{\partial x}\right|_{x=0} &= 0 = 3r^2 \mathrm{sign}(x-\xi)\big|_{x=0} + c_3 \\ &= -3\xi^2 + c_3 \\ M(G^{cf}(x,\xi))\big|_{x=l} &= 0 \rightarrow \\ \left.\frac{\partial^2 G^{cf}(x,\xi)}{\partial x^2}\right|_{x=l} &= 0 = 6r\big|_{x=l} + 6c_1 l + 2c_2 \\ &= 6(l-\xi) + 6c_1 l + 2c_2 \\ Q(G^{cf}(x,\xi))\big|_{x=l} &= 0 \rightarrow \\ \left.\frac{\partial^3 G^{cf}(x,\xi)}{\partial x^3}\right|_{x=l} &= 0 = 6\mathrm{sign}(x-\xi)\big|_{x=l} + 6c_1 \\ &= 6 + 6c_1 \end{aligned}$$

One easily gets

$$c_1 = -1, \; c_2 = 3\xi, \; c_3 = 3\xi^2, \; c_4 = -\xi^3$$

and hence, the following Green's function

$$G^{cf}(x,\xi) = \frac{1}{12EI}\left\{r^3 - x^3 + 3\xi x^2 + 3\xi^2 x - \xi^3\right\}$$

5.1.0.14 Exercise 12: Strain tensor of the elastostatic fundamental solution

Since differentiating the distance $r = |\mathbf{x} - \xi|$ with respect to the coordinates x_i gives in R^2 and in R^3

$$\frac{\partial r}{\partial x_i} = r_{,i} = \frac{x_i - \xi_i}{r}, \quad \frac{\partial^2 r}{\partial x_i \partial x_j} = r_{,ij} = \frac{1}{r}\left(\delta_{ij} - r_{,i}\, r_{,j}\right)$$

it is easy to determine

$$\frac{\partial u_i^{(k)}(\mathbf{x},\xi)}{\partial x_j} = \frac{1}{8\pi}\frac{1}{2\mu+\lambda}\frac{1}{r^2}\left[-(3+\frac{\lambda}{\mu})\delta_{ik} r_{,j} + (1+\frac{\lambda}{\mu})\left(\delta_{ij} r_{,k} + \delta_{kj} r_{,i} - 3 r_{,i}\, r_{,j}\, r_{,k}\right)\right] \quad \text{in } R^3$$

and then via the definition of the strain tensor

$$\begin{aligned}\varepsilon_{ij}^{(k)}(\mathbf{x},\xi) &= \frac{1}{2}\left(\frac{\partial u_i^{(k)}(\mathbf{x},\xi)}{\partial x_j} + \frac{\partial u_j^{(k)}(\mathbf{x},\xi)}{\partial x_i}\right) \\ &= \frac{1}{8\pi}\frac{1}{2\mu+\lambda}\frac{1}{r^2}\left[-\left(\delta_{ik} r_{,j} + \delta_{jk} r_{,i}\right) + (1+\frac{\lambda}{\mu})\left(\delta_{ij} r_{,k} - 3 r_{,i}\, r_{,j}\, r_{,k}\right)\right] \quad \text{in } R^3\end{aligned}$$

5.1.0.15 Exercise 13: Shape functions with the local coordinate $-1 \le \eta \le 1$

Since the restrictions (4.2) hold, the **linear shape functions**

$$N_1^2(\eta) = a_1\eta + b_1, \quad N_2^2(\eta) = a_2\eta + b_2$$

are defined by the two conditions for each of the shape functions

$$\begin{aligned}N_1^2(-1) &= -a_1 + b_1 = 1, \;\; N_1^2(+1) = a_1 + b_1 = 0 \\ N_2^2(-1) &= -a_2 + b_2 = 0, \;\; N_2^2(+1) = a_2 + b_2 = 1\end{aligned}$$

This gives easily for $-1 \le \eta \le 1$

$$N_1^2(\eta) = \frac{1}{2}(1-\eta), \quad N_2^2(\eta) = \frac{1}{2}(1+\eta)$$

For the three **quadratic shape functions**

$$N_1^3(\eta) = a_1\eta^2 + b_1\eta + c_1, \; N_2^3(\eta) = a_2\eta^2 + b_2\eta + c_2, \; N_3^3(\eta) = a_3\eta^2 + b_3\eta + c_3$$

the restrictions (4.2) give three conditions for each of them

$$\begin{aligned}N_1^3(-1) &= a_1 - b_1 + c_1 = 1, \; N_1^3(0) = c_1 = 0, \; N_1^3(1) = a_1 + b_1 + c_1 = 0 \\ N_2^3(-1) &= a_2 - b_2 + c_2 = 0, \; N_2^3(0) = c_2 = 1, \; N_2^3(1) = a_2 + b_2 + c_2 = 0 \\ N_3^3(-1) &= a_3 - b_3 + c_3 = 0, \; N_3^3(0) = c_3 = 0, \; N_3^3(1) = a_3 + b_3 + c_3 = 1\end{aligned}$$

and one obtains for $-1 \le \eta \le 1$

$$N_1^3(\eta) = \frac{1}{2}\eta(\eta-1), \; N_2^3(\eta) = 1 - \eta^2, \; N_3^3(\eta) = \frac{1}{2}\eta(\eta+1)$$

5.1.0.16 Exercise 14: Stationary heat conduction in a rectangular domain-constant shape functions

For the simplest discretization of the boundary, i.e., each side of the quadrilateral domain as an element, collocations at the middle nodes of these four elements, at $\mathbf{x}^1 = (l_1/2, 0)$, $\mathbf{x}^2 = (l_1, l_2/2)$, $\mathbf{x}^3 = (l_1/2, l_2)$, and $\mathbf{x}^4 = (0, l_2/2)$ yield from (4.55), (4.56), and (4.57), respectively, the following matrix elements of the algebraic systemrixelemente (4.49):

$$
\begin{aligned}
G_{11} &= G_{22} = G_{33} = G_{44} = 0 \\
G_{12} &= G_{14} = G_{32} = G_{34} = -\frac{1}{2\pi}\arctan(4) \approx -0.2110 \\
G_{13} &= G_{31} = -\frac{1}{\pi}\arctan(\frac{1}{4}) \approx -0.07798 \\
G_{21} &= G_{23} = G_{41} = G_{43} = -0.125 \ \text{ und } \ G_{24} = G_{42} = -0.25
\end{aligned}
$$

$$
\begin{aligned}
H_{11} &= H_{33} = \frac{1}{2}H_{22} = \frac{1}{2}H_{44} = \frac{1}{2\pi\lambda_0}(1+\ln(2)) \approx \frac{0.269473}{\lambda_0} \\
H_{12} &= H_{14} = H_{32} = H_{34} = \frac{1}{4\pi\lambda_0}\left\{4 - 2\ln(\frac{17}{16}) - \arctan(4)\right\} \approx \frac{0.203156}{\lambda_0} \\
H_{13} &= H_{31} = \frac{1}{4\pi\lambda_0}\left\{2 - \ln(\frac{17}{4}) - 8\arctan(\frac{1}{4})\right\} \approx -\frac{0.1119454}{\lambda_0} \\
H_{21} &= H_{23} = H_{41} = H_{43} = \frac{1}{4\pi\lambda_0}\left\{2 - \ln(2) - \frac{\pi}{2}\right\} \approx -\frac{0.0210}{\lambda_0} \\
H_{24} &= H_{42} = \frac{1}{4\pi\lambda_0}\left\{4 + 2\ln(2) - \pi\right\} \approx \frac{0.178627}{\lambda_0}
\end{aligned}
$$

Taking the boundary conditions $\Theta^1 = 0$, $\Theta^3 = \overline{\Theta}$, and $q_n^2 = q_n^4 = 0$ into account, and reordering the system with respect to known and unknown node values, gives the system

$$
\begin{bmatrix} G_{12} & G_{12} & -H_{11} & -H_{13} \\ 0.5 & G_{24} & -H_{21} & -H_{21} \\ G_{12} & G_{12} & -H_{13} & -H_{11} \\ G_{24} & 0.5 & -H_{21} & -H_{21} \end{bmatrix}
\begin{bmatrix} \Theta^2 \\ \Theta^4 \\ q_n^1 \\ q_n^3 \end{bmatrix}
= \overline{\Theta} \begin{bmatrix} -G_{13} \\ -G_{23} \\ -0.5 \\ -G_{23} \end{bmatrix}
$$

Subtraction of the forth from the second equationt gives $\Theta^2 = \Theta^4$ and then by adding these two equations

$$
\Theta^2 = \Theta^4 = \frac{-G_{23}}{0.5 + G_{24}}\overline{\Theta} \ + \frac{H_{21}}{0.5 + G_{24}}(q_n^1 + q_n^3) = \frac{1}{2}\overline{\Theta} \ + 4H_{21}(q_n^1 + q_n^3)
$$

Finally, subtracting and adding the first and third equation and taking the above result for $\Theta^2 = \Theta^4$ into account gives

$$
q_n^1 - q_n^3 = \frac{0.5 - G_{13}}{H_{13} - H_{11}}\overline{\Theta} \ \text{ and } \ q_n^1 + q_n^3 = \frac{0.5 + G_{13} + 2G_{12}}{H_{11} + H_{13} - 16G_{12}H_{12}}\bar{\Theta}
$$

Since $G_{13} + 2G_{12} = -\frac{1}{\pi}\arctan(\frac{1}{4}) - 2\frac{1}{2\pi}\arctan(4) = -\frac{1}{\pi}(\arctan(\frac{1}{4}) + \arctan(4)) = -\frac{1}{\pi}\frac{\pi}{2} = -0.5$, the nominator $0.5 + G_{13} + 2G_{12} = 0$ and, hence, $q_n^1 = -q_n^3$. This yields

$$\begin{aligned} \Theta^2 &= \Theta^4 = \frac{1}{2}\overline{\Theta} \\ q_n^1 - q_n^3 &= 2q_n^1 = \frac{0.5 - G_{13}}{H_{13} - H_{11}}\overline{\Theta} \Longrightarrow q_n^1 = -q_n^3 = \frac{0.5 - G_{13}}{2(H_{13} - H_{11})} \approx -0.757\lambda_0\overline{\Theta} \end{aligned}$$

The exact solution gives for these node points the same value for the temperature $\Theta(l_1, l_2/2) = \Theta(0, l_2/2) = 0.5\overline{\Theta}$, while with $n_2(l_1/2, 0) = -1$, $n_2(l_1/2, 0) = 1$, and $l_2 = 2m$ the heat flux is there exactly $q_n^1 = q_n(l_1/2, 0) = -\lambda_0\bar{\Theta}/l_2 = -0.5\lambda_0\overline{\Theta} = -q_n^3 = -q_n(l_1/2, l_2)$ which means about 50% error. This not at all astonishing since physically wrong elementwise constant approximations have been used to model the correctly continous temperature field.

5.1.0.17 Exercise 15: Stationary heat conduction in a rectangular domain-linear shape functions

When applying the simplest possible discretization of the boundary, i.e., taking each of the 4 sides of the rectangular domain as one boundary element, using linear shape functions for the temperature Θ and the heat flux q_n, and performing collocation at the nodes $\mathbf{x}^1 = (0, 0)$, $\mathbf{x}^2 = (l_1, 0)$, $\mathbf{x}^3 = (l_1, l_2)$, and $\mathbf{x}^4 = (0, l_2)$, (here, the 4 corners of the domain), one obtains the system

$$\sum_{e=1}^{4} \left(\frac{1}{4}\delta_{je} + G_{je}\right)\Theta^e = \sum_{e=1}^{4} \left(H_{je}^2 q_n^{2e} + H_{je}^1 q_n^{1e}\right) \tag{5.9}$$

where here, since all node points are corner points, $q_n^{2e} \neq q_n^{1e+1}$ for all 4 elemente Γ^e.

From (4.69) and (4.70), respectively, result the following matrix elements

$$\begin{aligned} G_{11} &= G_{22} = G_{33} = G_{44} = 0 \\ G_{21} &= G_{12} = G_{34} = G_{43} = \frac{1}{8\pi}\left[\ln(5) - 4\arctan(2)\right] \\ G_{31} &= G_{13} = G_{24} = G_{42} = \frac{1}{8\pi}\left[4\ln(4) - 5\ln(5)\right] \\ G_{41} &= G_{14} = G_{23} = G_{32} = \frac{1}{2\pi}\left[\ln(\frac{5}{4}) - \arctan(\frac{1}{2})\right] \end{aligned}$$

while from (4.71) and (4.72), respectively, the elements of H_{je}^1 and H_{je}^2 can be evaluated corresponding to the location of the collocation point x_i^j from (4.73) to (4.76) and (4.78) to (4.77),respectively: $(l_1 = l_3 = l = 1m, l_2 = l_4 = 2l = 2m)$

$$
\begin{aligned}
H^1_{11} &= H^1_{33} = \frac{3l_1}{8\pi\lambda_0},\ H^1_{22} = H^1_{44} = \frac{3l_2}{4\pi\lambda_0} \\
H^1_{21} &= H^1_{43} = \frac{l_1}{8\pi\lambda_0};\ H^1_{12} = H^1_{34} = \frac{-l_2}{32\pi\lambda_0}\left[3\ln(5) - 4\ln(4) - 12 + 8\arctan(2)\right] \\
H^1_{31} &= H^1_{13} = \frac{-l_1}{8\pi\lambda_0}\left[5\ln(5) - 4\ln(4) - 1\right];\ H^1_{24} = H^1_{42} = \frac{-l_2}{32\pi\lambda_0}\left[5\ln(5) - 4\ln(4) - 4\right] \\
H^1_{41} &= H^1_{23} = \frac{-l_1}{8\pi\lambda_0}\left[4\ln(4) - 3\ln(5) - 3 + 8\arctan(\frac{1}{2})\right];\ H^1_{14} = H^1_{32} = \frac{l_2}{8\pi\lambda_0}
\end{aligned}
$$

$$
\begin{aligned}
H^2_{11} &= H^2_{33} = H^1_{21},\quad H^2_{22} = H^2_{44} = H^1_{14} \\
H^2_{21} &= H^2_{43} = H^1_{11};\quad H^2_{12} = H^2_{34} = H^1_{24} \\
H^2_{31} &= H^2_{13} = H^1_{41};\quad H^2_{24} = H^2_{42} = H^1_{34} \\
H^2_{41} &= H^2_{23} = H^1_{13};\quad H^2_{14} = H^2_{32} = H^1_{22}
\end{aligned}
$$

With that, and using the abbreviations $h_1 = 2\ln(4) + 2 - \frac{5}{2}\ln(5)$, $h_2 = -4\ln(4) + 3\ln(5) + 3 - 8\arctan(\frac{1}{2})$, $h_3 = 4\ln(4) + 1 - 5\ln(5)$, $h_4 = -\frac{3}{2}\ln(5) + 2\ln(4) + 6 - 4\arctan(2)$ and $g_1 = \frac{1}{4}\ln(5) - \arctan(2)$, $g_2 = \ln(4) - \frac{5}{4}\ln(5)$, $g_3 = \ln(\frac{5}{4}) - \arctan(\frac{1}{2})$ the system (5.9) becomes

$$
\frac{l}{8\pi\lambda_0}\left(\begin{bmatrix} 1 & h_1 & h_2 & 6 \\ 3 & 2 & h_3 & h_4 \\ h_2 & 6 & 1 & h_1 \\ h_3 & h_4 & 3 & 2 \end{bmatrix}\begin{bmatrix} q_n^{21} \\ q_n^{22} \\ q_n^{23} \\ q_n^{24} \end{bmatrix} + \begin{bmatrix} 3 & h_4 & h_3 & 2 \\ 1 & 6 & h_2 & h_1 \\ h_3 & 2 & 3 & h_4 \\ h_2 & h_1 & 1 & 6 \end{bmatrix}\begin{bmatrix} q_n^{11} \\ q_n^{12} \\ q_n^{13} \\ q_n^{14} \end{bmatrix}\right) =
$$

$$
= \frac{1}{2\pi}\begin{bmatrix} \frac{\pi}{2} & g_1 & g_2 & g_3 \\ g_1 & \frac{\pi}{2} & g_3 & g_2 \\ g_2 & g_3 & \frac{\pi}{2} & g_1 \\ g_3 & g_2 & g_1 & \frac{\pi}{2} \end{bmatrix}\begin{bmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \\ \Theta^4 \end{bmatrix}
$$

Now, incorporating the actual boundary conditions of the problem, i.e.,

$$\Theta^1 = \Theta^2 = 0 \ \text{ and } \ \Theta^3 = \Theta^4 = \bar{\Theta}$$

and

$$q_n^{22} = q_n^{24} = q_n^{12} = q_n^{14} = 0$$

reduces the system to:

$$
\frac{l}{8\pi\lambda_0}\begin{bmatrix} 3 & 1 & h_3 & h_2 \\ 1 & 3 & h_2 & h_3 \\ h_3 & h_2 & 3 & 1 \\ h_2 & h_3 & 1 & 3 \end{bmatrix}\begin{bmatrix} q_n^{11} \\ q_n^{21} \\ q_n^{13} \\ q_n^{23} \end{bmatrix} = \frac{\bar{\Theta}}{2\pi}\begin{bmatrix} g_2 + g_3 \\ g_2 + g_3 \\ \frac{\pi}{2} + g_1 \\ \frac{\pi}{2} + g_1 \end{bmatrix}
$$

Subtraction of the second from the first and of the forth from the third equation yield $q_n^{11} = q_n^{21}$ and $q_n^{13} = q_n^{23}$, so that only two equations remain

$$\begin{aligned} q_n^{11} + \left[1 - \frac{1}{2}\ln(5) - 2\arctan(\frac{1}{2})\right] q_n^{13} &= -\frac{\bar{\Theta}}{4l}\lambda_0 \left[\ln(5) + 4\arctan(\frac{1}{2})\right] \\ \left[1 - \frac{1}{2}\ln(5) - 2\arctan(\frac{1}{2})\right] q_n^{11} + q_n^{13} &= \frac{\bar{\Theta}}{4l}\lambda_0 \left[2\pi + \ln(5) - 4\arctan(2)\right] \end{aligned}$$

From that, one obtains with $\arctan(\frac{1}{2}) + \arctan(2) = \frac{\pi}{2}$ the exact boundary values for the heat flux

$$q_n^{13} = q_n^{23} = \frac{\bar{\Theta}}{2l}\lambda_0 \text{ and } q_n^{11} = q_n^{21} = -\frac{\bar{\Theta}}{2l}\lambda_0$$

The temperature distribution is by the linear shape function on the element Γ^2,i.e., on $x_1 = x_1^{12} = x_1^{22} = l_1$, $0 = x_2^{12} \le x_2 \le x_2^{22} = l_2$, with $\eta = x_2/l_2$ and the boundary conditions $\Theta^{12} = 0$ and $\Theta^{22} = \overline{\Theta}$ exactly described

$$\Theta(x_i^2(\eta)) = \Theta^{12}(1-\eta) + \Theta^{22}\eta = \bar{\Theta}\frac{x_2}{l_2}$$

This is also true on Γ^4, i.e., $x_1 = x_1^{14} = x_1^{24} = 0$, $0 = x_2^{24} \le x_2 \le x_2^{14} = l_2$, with $\eta = \frac{x_2 - l_2}{l_2}$ and the boundary conditions $\Theta^{24} = 0$ and $\Theta^{14} = \bar{\Theta}$ by

$$\Theta(x_i^4(\eta)) = \Theta^{14}(1-\eta) + \Theta^{24}\eta = \bar{\Theta}(1 - \frac{x_2 - l_2}{l_2}) = \bar{\Theta}\frac{x_2}{l_2}.$$

From (4.66), the temperature at internal points $\mathbf{x}^j$ is in this case , i.e., with the prescribed boundary values $\Theta^1 = \Theta^{11} = \Theta^{24} = 0$, $\Theta^2 = \Theta^{12} = \Theta^{21} = 0$, $\Theta^3 = \Theta^{22} = \Theta^{13} = \bar{\Theta}$, $\Theta^4 = \Theta^{23} = \Theta^{14} = \overline{\Theta}$, and $q_n^{22} = q_n^{24} = q_n^{12} = q_n^{14} = 0$ and the just determined boundary reactions $q_n^{13} = q_n^{23} = \frac{\bar{\Theta}}{2l}\lambda_0$ and $q_n^{11} = q_n^{21} = -\frac{\bar{\Theta}}{2l}\lambda_0$ given by

$$\begin{aligned} \Theta(\mathbf{x}^j) &= \frac{\bar{\Theta}}{2l}\lambda_0 \int_{\eta=0}^{1} \left\{ \begin{array}{c} \Theta^*(\mathbf{x}^3(\eta),\mathbf{x}^j)(1-\eta)l_3 - \Theta^*(\mathbf{x}^1(\eta),\mathbf{x}^j)(1-\eta)l_1 \\ +\Theta^*(\mathbf{x}^3(\eta),\mathbf{x}^j)\eta l_3 \; - \Theta^*(\mathbf{x}^1(\eta),\mathbf{x}^j)\eta l_1 \end{array} \right\} d\eta \\ &\quad -\bar{\Theta} \int_{\eta=0}^{1} \left\{ \begin{array}{c} \left[q_n^*(\mathbf{x}^3(\eta),\mathbf{x}^j)l_3 + q_n^*(\mathbf{x}^4(\eta),\mathbf{x}^j)l_4\right](1-\eta) \\ + \left[q_n^*(\mathbf{x}^2(\eta),\mathbf{x}^j)l_2 + q_n^*(\mathbf{x}^3(\eta),\mathbf{x}^j)l_3\right]\eta \end{array} \right\} d\eta \\ &= \frac{\bar{\Theta}}{2l}\lambda_0 \int_{\eta=0}^{1} \left\{\Theta^*(\mathbf{x}^3(\eta),\mathbf{x}^j)l_3 - \Theta^*(\mathbf{x}^1(\eta),\mathbf{x}^j)l_1\right\} d\eta \\ &\quad -\bar{\Theta} \int_{\eta=0}^{1} \left\{ \begin{array}{c} \left[q_n^*(\mathbf{x}^3(\eta),\mathbf{x}^j)l_3 + q_n^*(\mathbf{x}^4(\eta),\mathbf{x}^j)l_4\right] \\ + \left[q_n^*(\mathbf{x}^2(\eta),\mathbf{x}^j)l_2 - q_n^*(\mathbf{x}^4(\eta),\mathbf{x}^j)l_4\right]\eta \end{array} \right\} d\eta \end{aligned} \tag{5.10}$$

where as in (4.57)

$$\int_{\eta=0}^{1} \Theta^*(\mathbf{x}^e(\eta),\mathbf{x}^j)l_e \, d\eta = \frac{-1}{4\pi\lambda_0} \left[\begin{array}{c} (l_e\eta + \mathbf{r}^{ej}\cdot\mathbf{t}^e)\ln\left(\eta^2 + \frac{2}{l_e}\mathbf{r}^{ej}\cdot\mathbf{t}^e\eta + \frac{|\mathbf{x}^{1e}-\mathbf{x}^j|^2}{l_e^2}\right) \\ -2l_e\eta + 2\mid\mathbf{r}^{ej}\cdot\mathbf{n}^e\mid\arctan\frac{l_e\eta + \mathbf{r}^{ej}\cdot\mathbf{t}^e}{|\mathbf{r}^{ej}\cdot\mathbf{n}^e|} \end{array} \right]_{\eta=0}^{1} \tag{5.11}$$

and $r_{ej} =\mid \mathbf{x}^{1e} - \mathbf{x}^j \mid$ means the distance between the observation point $\mathbf{x}^j$ and the initial point $\mathbf{x}^{1e}$ of the element Γ^e. With the angle $\alpha_{ej} = \sphericalangle(\mathbf{r}^{ej}, \mathbf{t}^e)$, between $\mathbf{r}^{ej}$ and the tangential unit vector $\mathbf{t}^e$ along this element holds

$$\begin{aligned} (x_i^{1e} - x_i^j)t_i^e &= -r_{ej}\cos\alpha_{ej}, \quad n_i^e(x_i^{1e} - x_i^j) = r_{ej}\sin\alpha_{ej} \\ l_e^2 + r_{ej}^2 - 2l_e r_{ej}\cos\alpha_{ej} &= r_{e+1,j}^2 \end{aligned} \tag{5.12}$$

where for rectangular domains, as here, is especially

$$\begin{aligned} l_e - r_{ej}\cos\alpha_{ej} &= r_{e+1,j}\sin\alpha_{e+1,j} \\ r_{e+1,j}\cos\alpha_{e+1,j} &= r_{ej}\sin\alpha_{ej} \qquad \text{für } e = 1,2,3,4 \text{ mit } e = 5 \triangleq 1 \end{aligned} \tag{5.13}$$

With that, the following integral can be evaluated as

$$\begin{aligned} \int\limits_{\eta=0}^{1} \Theta^*(\mathbf{x}^e(\eta), \mathbf{x}^j) l_e \, d\eta &= \frac{-1}{4\pi\lambda_0} \left[\begin{array}{c} r_{e+1,j}\sin\alpha_{e+1,j}\ln\left(\frac{r_{e+1,j}^2}{l_e^2}\right) - 2l_e + r_{ej}\cos\alpha_{ej}\ln\left(\frac{r_{e,j}^2}{l_e^2}\right) \\ +2r_{ej}\sin\alpha_{ej}\left(\arctan\frac{r_{e+1,j}\sin\alpha_{e+1,j}}{r_{ej}\sin\alpha_{ej}} + \arctan(\cot\alpha_{ej})\right) \end{array} \right] \\ &= \frac{-1}{4\pi\lambda_0} \left[\begin{array}{c} r_{e+1,j}\sin\alpha_{e+1,j}\ln\left(\frac{r_{e+1,j}^2}{l_e^2}\right) - 2l_e + r_{ej}\cos\alpha_{ej}\ln\left(\frac{r_{e,j}^2}{l_e^2}\right) \\ +2r_{ej}\sin\alpha_{ej}\left(\alpha_{e+1,j} + \frac{\pi}{2} - \alpha_{ej}\right) \end{array} \right] \end{aligned}$$

From (4.56) is

$$\begin{aligned} \int\limits_{\eta=0}^{1} q_n^*(\mathbf{x}^e(\eta), \mathbf{x}^j) l_e \, d\eta &= \frac{-1}{2\pi}\text{sign}\left(\mathbf{n}^e \cdot \mathbf{r}^{ej}\right)\left\{\arctan\frac{l_e + \mathbf{t}^e \cdot \mathbf{r}^{ej}}{\mid \mathbf{n}^e \cdot \mathbf{r}^{ej} \mid} - \arctan\frac{\mathbf{t}^e \cdot \mathbf{r}^{ej}}{\mid \mathbf{n}^e \cdot \mathbf{r}^{ej} \mid}\right\} \\ &= \frac{-1}{2\pi}\text{sign}\left(\mathbf{n}^e \cdot \mathbf{r}^{ej}\right)\left\{\arctan\frac{r_{e+1,j}\sin\alpha_{e+1,j}}{r_{ej}\sin\alpha_{ej}} - \arctan(\cot\alpha_{ej})\right\} \end{aligned}$$

and, especially for the actual rectangular domain with $\alpha_{ej} \leq 0.5\pi$

$$\int\limits_{\eta=0}^{1} q_n^*(\mathbf{x}^e(\eta), \mathbf{x}^j) l_e \, d\eta = \frac{-1}{2\pi}\left\{\alpha_{e+1,j} + \frac{\pi}{2} - \alpha_{ej}\right\}$$

and, moreover, from (4.69)

$$\begin{aligned} \int\limits_{\eta=0}^{1} q_n^*(\mathbf{x}^e(\eta), \mathbf{x}^j)\eta l_e \, d\eta &= \frac{-\mathbf{n}^e \cdot \mathbf{r}^{ej}}{2\pi l_e} \left\{ \begin{array}{c} \frac{1}{2}\ln\left(1 + 2\frac{l_e \mathbf{t}^e \cdot \mathbf{r}^{ej}}{|\mathbf{x}^{1e} - \mathbf{x}^j|^2} + \frac{l_e^2}{|\mathbf{x}^{1e} - \mathbf{x}^j|^2}\right) \\ -\frac{\mathbf{t}^e \cdot \mathbf{r}^{ej}}{|\mathbf{n}^e \cdot \mathbf{r}^{ej}|}\left(\arctan\frac{l_e + \mathbf{t}^e \cdot \mathbf{r}^{ej}}{|\mathbf{n}^e \cdot \mathbf{r}^{ej}|} - \arctan\frac{\mathbf{t}^e \cdot \mathbf{r}^{ej}}{|\mathbf{n}^e \cdot \mathbf{r}^{ej}|}\right) \end{array} \right\} \\ &= \frac{-1}{2\pi l_e} \left\{ \begin{array}{c} \frac{1}{2} r_{ej}\sin\alpha_{ej}\ln\left(\frac{r_{e+1,j}^2}{r_{e,j}^2}\right) + \\ \text{sign}\left(\mathbf{n}^e \cdot \mathbf{r}^{ej}\right) r_{ej}\cos\alpha_{ej} \left(\begin{array}{c} \arctan\frac{r_{e+1,j}\sin\alpha_{e+1,j}}{r_{ej}\sin\alpha_{ej}} \\ +\frac{\pi}{2} - \alpha_{ej} \end{array} \right) \end{array} \right\} \end{aligned}$$

and especially for rectangular domains

$$\int\limits_{\eta=0}^{1} q_n^*(\mathbf{x}^e(\eta), \mathbf{x}^j)\eta l_e \, d\eta = \frac{-1}{2\pi l_e}\left\{\frac{1}{2} r_{ej}\sin\alpha_{ej}\ln\left(\frac{r_{e+1,j}^2}{r_{e,j}^2}\right) + r_{ej}\cos\alpha_{ej}\left(\alpha_{e+1,j} + \frac{\pi}{2} - \alpha_{ej}\right)\right\}$$

Finally, one obtains for (5.10) in the actual rectangular domain with $l_1 = l_3 = l$ and $l_2 = l_4 = 2l$

$$\Theta(\mathbf{x}^j) = \frac{-\bar{\Theta}}{8\pi l}\left\{\begin{array}{c}\left[\begin{array}{c} r_{4j}\sin\alpha_{4j}\ln\left(\frac{r_{4j}^2}{l^2}\right) - 2l + r_{3j}\cos\alpha_{3j}\ln\left(\frac{r_{3j}^2}{l^2}\right) \\ +2r_{3j}\sin\alpha_{3j}\left(\alpha_{4j}+\frac{\pi}{2}-\alpha_{3j}\right)\end{array}\right] \\ -\left[\begin{array}{c} r_{2j}\sin\alpha_{2j}\ln\left(\frac{r_{2j}^2}{l^2}\right) - 2l + r_{1j}\cos\alpha_{1j}\ln\left(\frac{r_{1j}^2}{l^2}\right) \\ +2r_{1j}\sin\alpha_{1j}\left(\alpha_{2j}+\frac{\pi}{2}-\alpha_{1j}\right)\end{array}\right]\end{array}\right\}$$
$$+\frac{\bar{\Theta}}{2\pi}\left\{\begin{array}{c}\left[\alpha_{4j}+\frac{\pi}{2}-\alpha_{3j}\right]+\left[\alpha_{1j}+\frac{\pi}{2}-\alpha_{4j}\right] \\ +\frac{1}{2l}\left[\frac{1}{2}r_{2j}\sin\alpha_{2j}\ln\left(\frac{r_{3j}^2}{r_{2j}^2}\right)+r_{2j}\cos\alpha_{2j}\left(\alpha_{3j}+\frac{\pi}{2}-\alpha_{2j}\right)\right] \\ -\frac{1}{2l}\left[\frac{1}{2}r_{4j}\sin\alpha_{4j}\ln\left(\frac{r_{1j}^2}{r_{4j}^2}\right)+r_{4j}\cos\alpha_{4j}\left(\alpha_{1j}+\frac{\pi}{2}-\alpha_{4j}\right)\right]\end{array}\right\}$$

Taking into account that $x_2^j = r_{1j}\sin\alpha_{1j}$ and $2l - x_2^j = r_{3j}\sin\alpha_{3j}$ as well as (5.13) hold, this is reduced to

$$\Theta(\mathbf{x}^j) = \frac{\bar{\Theta}}{2l}x_2^j$$

the exact solution for internal points.

5.2 B: Analytic integration of singular boundary integrals

5.2.1 Analytic integration in the case of logarithmic kernels

In R^2, kernels containing $\ln(r/c)$ are weakly singular and can be represented on a straight boundary element modelled by the linear approximation (4.1) $x_i^e(\eta) = x_i^{1e}(1-\eta) + x_i^{2e}\eta$ as

$$\ln(\frac{r}{c}) = \frac{1}{2}\ln\left[\left(x_1^{1e}(1-\eta)+x_1^{2e}\eta-\xi_1\right)^2+\left(x_2^{1e}(1-\eta)+x_2^{2e}\eta-\xi_2\right)^2\right]-\ln(c) \qquad (5.14)$$

For an approximation of the boundary state $\Phi(\mathbf{x})$ in Γ_e by a constant value $\Phi(x_i^e(\eta)) = \Phi^e$ and a collocation at its respective node, i.e., at the middle of the element $\xi_i = \frac{1}{2}(x_i^{1e}+x_i^{2e})$, an integral with a logarithmically singular kernel over this element can be evaluated exactly

$$\begin{aligned}\int_{\Gamma_e}\ln(\frac{r(\mathbf{x},\xi)}{c})\Phi(\mathbf{x})d\Gamma_{\mathbf{x}} &= \Phi^e\left[\frac{l_e}{2}\int_{\eta=0}^{1}\ln\left\{(\frac{1}{2}-\eta)^2\left[(x_1^{2e}-x_1^{1e})^2+(x_2^{2e}-x_2^{1e})^2\right]\right\}d\eta - l_e\ln(c)\right] \\ &= \Phi^e\left[\frac{l_e}{2}\int_{\eta=0}^{1}\left\{\ln(\frac{1}{2}-\eta)^2+\ln l_e^2\right\}d\eta - l_e\ln(c)\right] \\ &= \Phi^e\left[\frac{l_e}{2}\left\{2(\ln\frac{1}{2}-1)+2\ln l_e\right\}-l_e\ln(c)\right] \\ &= \Phi^e l_e\left\{\ln(\frac{l_e}{2c})-1\right\} \qquad (5.15) \\ &= -\Phi^e l_e(\ln 2+1) \text{ for } c = l_e \text{ (see the Remark below)} \qquad (5.16)\end{aligned}$$

Remark: The argument of the ln-function has to be dimensionless and, hence, the constant factor c in the above integral kernel has to represent a distance, but can be arbitrarily chosen. In order to avoid that for smaller elements, i.e., $l_e \to 0$, the term $\ln(\frac{l_e}{2c})$ becomes dominant due to $\lim_{l_e \to 0} \left\{ \ln(\frac{l_e}{2c}) \right\} \to -\infty$, the factor c should be chosen problem orientated, e,g., as $c = l_e$.

When the state function is **linearly** approximated

$$\Phi(x_i^e(\eta)) = \Phi^{1e}(1-\eta) + \Phi^{2e}\eta$$

and the collocation point ξ is placed at the initial node of the element, i.e., $\xi_i = x_i^{1e}$, the square of the distance between this collocation point and the integration points is $r^2 = l_e^2\eta^2$. Then, the integral with a logarithmic kernel becomes (c is an arbitrary constant distance, e.g., $c = l_e$)

$$\begin{aligned} \int\limits_{\Gamma_e} \ln(\frac{r(\mathbf{x},\xi)}{c})\Phi(\mathbf{x})d\Gamma_{\mathbf{x}} &= l_e \int\limits_{\eta=0}^{1} \left\{ \left[\ln\eta + \ln(\frac{l_e}{c}) \right] \left[\Phi^{1e}(1-\eta) + \Phi^{2e}\eta \right] \right\} d\eta \\ &= \frac{l_e}{4} \left\{ \Phi^{1e}(2\ln(\frac{l_e}{c}) - 3) + \Phi^{2e}(2\ln(\frac{l_e}{c}) - 1) \right\} \\ &= -\frac{l_e}{4} \left\{ 3\Phi^{1e} + \Phi^{2e} \right\} \quad \text{for } c = l_e \end{aligned} \tag{5.17}$$

while for a collocation at the end node of the element, i.e., at $\xi_i = x_i^{2e}$, the square of the distance between this collocation point and the integration points is $r^2 = l_e^2(1-\eta)^2$ and the integral becomes

$$\begin{aligned} \int\limits_{\Gamma_e} \ln(\frac{r(\mathbf{x},\xi)}{c})\Phi(\mathbf{x})d\Gamma_{\mathbf{x}} &= l_e \int\limits_{\eta=0}^{1} \left\{ \left[\ln(1-\eta) + \ln(\frac{l_e}{c}) \right] \left[\Phi^{1e}(1-\eta) + \Phi^{2e}\eta \right] \right\} d\eta \\ &= \frac{l_e}{4} \left\{ \Phi^{1e} \left[2\ln(\frac{l_e}{c}) - 1 \right] + \Phi^{2e} \left[2\ln(\frac{l_e}{c}) - 3 \right] \right\} \\ &= -\frac{l_e}{4} \left\{ \Phi^{1e} + 3\Phi^{2e} \right\} \quad \text{for } c = l_e \end{aligned} \tag{5.18}$$

In the case of a **quadratic** approximation of the state function

$$\Phi(x_i^e(\eta)) = \Phi^{1e}(1-3\eta+2\eta^2) + \Phi^{2e}(4\eta-4\eta^2) + \Phi^{3e}(2\eta^2-\eta)$$

and collocation at the initial node of the element, i.e., at $\xi_i = x_i^{1e}$, the square of the distance between this collocation point and the integration points is $r^2 = l_e^2\eta^2$ and the integral becomes

$$\begin{aligned} \int\limits_{\Gamma_e} \ln(\frac{r(\mathbf{x},\xi)}{c})\Phi(\mathbf{x})d\Gamma_{\mathbf{x}} &= l_e \int\limits_{\eta=0}^{1} \left\{ \left[\ln\eta + \ln(\frac{l_e}{c}) \right] \begin{bmatrix} \Phi^{1e}(1-3\eta+2\eta^2)+ \\ \Phi^{2e}(4\eta-4\eta^2) \\ +\Phi^{3e}(2\eta^2-\eta) \end{bmatrix} \right\} d\eta \\ &= \frac{l_e}{6} \left\{ \Phi^{1e}(\ln(\frac{l_e}{c}) - \frac{17}{6}) + \Phi^{2e}(4\ln(\frac{l_e}{c}) - \frac{10}{3}) + \Phi^{3e}(\ln(\frac{l_e}{c}) + \frac{1}{6}) \right\} \\ &= -\frac{l_e}{36} \left\{ 17\Phi^{1e} + 20\Phi^{2e} - \Phi^{3e} \right\} \quad \text{for } c = l_e \end{aligned} \tag{5.19}$$

while for collocation at the middle point, i.e., at $\xi_i = x_i^{2e}$ gives $r^2 = l_e^2(\frac{1}{2} - \eta)^2$and

$$\begin{aligned}
\int\limits_{\Gamma_e} \ln(\frac{r(\mathbf{x},\xi)}{c})\Phi(\mathbf{x})d\Gamma_{\mathbf{x}} &= \frac{l_e}{2}\int\limits_{\eta=0}^{1}\left\{\left[\ln(\frac{1}{2}-\eta)^2 + 2\ln(\frac{l_e}{c})\right]\left[\begin{array}{c}\Phi^{1e}(1-3\eta+2\eta^2)+\\ \Phi^{2e}(4\eta-4\eta^2)\\ +\Phi^{3e}(2\eta^2-\eta)\end{array}\right]\right\}d\eta \\
&= \frac{l_e}{6}\left\{\left(\Phi^{1e}+\Phi^{3e}\right)(\ln(\frac{l_e}{2c}) - \frac{1}{3}) + 4\Phi^{2e}(\ln\frac{l_e}{2c} - \frac{4}{3})\right\} \qquad (5.20) \\
&= -\frac{l_e}{6}\left\{\left(\Phi^{1e}+\Phi^{3e}\right)(\ln 2 + \frac{1}{3}) + 4\Phi^{2e}(\ln 2 + \frac{4}{3})\right\} \quad \text{for } c = l_e
\end{aligned}$$

and for collokation at the end point of the element, i.e., at $\xi_i = x_i^{3e}$ gives $r^2 = l_e^2(1-\eta)^2$ and results in

$$\begin{aligned}
\int\limits_{\Gamma_e} \ln(\frac{r(\mathbf{x},\xi)}{c})\Phi(\mathbf{x})d\Gamma_{\mathbf{x}} &= l_e\int\limits_{\eta=0}^{1}\left\{\left[\ln(1-\eta) + \ln(\frac{l_e}{c})\right]\left[\begin{array}{c}\Phi^{1e}(1-3\eta+2\eta^2)+\\ \Phi^{2e}(4\eta-4\eta^2)\\ +\Phi^{3e}(2\eta^2-\eta)\end{array}\right]\right\}d\eta \\
&= \frac{l_e}{6}\left\{\Phi^{1e}\left[\ln(\frac{l_e}{c}) + \frac{1}{6}\right] + \Phi^{2e}\left[4\ln(\frac{l_e}{c}) - \frac{10}{3}\right] + \Phi^{3e}\left[\ln(\frac{l_e}{c}) - \frac{17}{6}\right]\right\} \\
&= -\frac{l_e}{36}\left\{-\Phi^{1e} + 20\Phi^{2e} + 17\Phi^{3e}\right\} \quad \text{for } c = l_e \qquad (5.21)
\end{aligned}$$

5.2.2 Analytic integration in the case of (1/r)- kernels

In many cases, integral kernels contain $r_{,s}/r$ which are strongly singular and, hence, the respective boundary integral has to be evaluated in the Cauchy principal value sense.

For a linear geometric approximation (4.1), i.e., for a straight boundary element with $x_i^e(\eta) = x_i^{1e}(1-\eta) + x_i^{2e}\eta$, the tangential unit vector is given as

$$t_i = \frac{dx_i^e}{ds} / \mid \frac{dx_i^e}{ds} \mid = \frac{dx_i^e}{d\eta} / \mid \frac{dx_i^e}{d\eta} \mid = \frac{x_i^{2e} - x_i^{1e}}{l_e} \qquad (5.22)$$

so that the distance between a collocation point ξ and the integration point on the element

$$r = \sqrt{\left[x_1^{1e}(1-\eta) + x_1^{2e}\eta - \xi_1\right]^2 + \left[x_2^{1e}(1-\eta) + x_2^{2e}\eta - \xi_2\right]^2} \qquad (5.23)$$

is found to have as tangential derivative

$$r_{,s} = r_{,i}t_i = \frac{\left[x_1^{1e}(1-\eta) + x_1^{2e}\eta - \xi_1, x_2^{1e}(1-\eta) + x_2^{2e}\eta - \xi_2\right]}{\sqrt{\left[x_1^{1e}(1-\eta) + x_1^{2e}\eta - \xi_1\right]^2 + \left[x_2^{1e}(1-\eta) + x_2^{2e}\eta - \xi_2\right]^2}} \cdot \frac{1}{l_e}\left[\begin{array}{c} x_1^{2e} - x_1^{1e} \\ x_2^{2e} - x_2^{1e}\end{array}\right] \qquad (5.24)$$

This allows to represent the singular kernel as

$$\frac{r_{,s}}{r} = \frac{\left[x_1^{1e}(1-\eta) + x_1^{2e}\eta - \xi_1, x_2^{1e}(1-\eta) + x_2^{2e}\eta - \xi_2\right]}{\left[x_1^{1e}(1-\eta) + x_1^{2e}\eta - \xi_1\right]^2 + \left[x_2^{1e}(1-\eta) + x_2^{2e}\eta - \xi_2\right]^2} \cdot \frac{1}{l_e}\left[\begin{array}{c} x_1^{2e} - x_1^{1e} \\ x_2^{2e} - x_2^{1e}\end{array}\right] \qquad (5.25)$$

and gives with a constant shape function for the state function and, accordingly, a collocation at the middle node of the element, i.e., at $\xi_i = \frac{1}{2}(x_i^{1e} + x_i^{2e})$, as distance (5.23) between the collocation point and the integration points $r = l_e \mid \frac{1}{2} - \eta \mid$. Finally, the strongly singular boundary integral can be evaluated to

$$\begin{aligned}
\int_{\Gamma_e} \frac{r_{,s}}{r}\Phi(\mathbf{x})d\Gamma_{\mathbf{x}} &= \Phi^e l_e \lim_{\varepsilon\to 0}\left(\int_{\eta=0}^{\frac{1}{2}-\varepsilon} + \int_{\eta=\frac{1}{2}+\varepsilon}^{1}\right) \frac{[x_1^{1e}-x_1^{2e}, x_2^{1e}-x_2^{2e}]\,(\frac{1}{2}-\eta)}{l_e^2 \mid \frac{1}{2}-\eta \mid^2}\frac{1}{l_e}\left[\begin{array}{c} x_1^{2e}-x_1^{1e} \\ x_2^{2e}-x_2^{1e} \end{array}\right] d\eta \\
&= \Phi^e \lim_{\varepsilon\to 0}\left(\int_{\eta=0}^{\frac{1}{2}-\varepsilon} + \int_{\eta=\frac{1}{2}+\varepsilon}^{1}\right)\frac{1}{\eta-\frac{1}{2}}d\eta \\
&= \Phi^e \lim_{\varepsilon\to 0}\left(\ln\varepsilon - \ln\frac{1}{2} + \ln\frac{1}{2} - \ln\varepsilon\right) = 0
\end{aligned} \tag{5.26}$$

For a linear approximation of the state function

$$\Phi(x_i^e(\eta)) = \Phi^{1e}(1-\eta) + \Phi^{2e}\eta$$

and collocation at the initial node of the element, i.e., at $\xi_i = x_i^{1e}$ one obtains for the distance between the integration points on the element Γ_e and that collocation point $r_e = l_e\eta$. At the same time, that collocation point also coincides with the end point of the neighbour element Γ_{e-1}, i.e., $\xi_i = x_i^{2e-1}$, which gives for the respective distance $r_{e-1} = l_{e-1}(1-\eta)$. In this case, the Cauchy principal value has to be evaluated with $\lim_{\varepsilon\to 0}$ by considering both neighbour elements Γ_{e-1} and Γ_e (regarding that $\Phi^{2e-1} = \Phi^{1e}$):

$$\begin{aligned}
\left(\int_{\Gamma_{e-1}} + \int_{\Gamma_e}\right)\frac{r_{,s}}{r}\Phi(\mathbf{x})d\Gamma_{\mathbf{x}} &= \int_{\eta=0}^{1-\varepsilon} \frac{[x_1^{1e-1}-x_1^{2e-1}, x_2^{1e-1}-x_2^{2e}]}{l_{e-1}^2(1-\eta)}\left[\begin{array}{c} x_1^{2e-1}-x_1^{1e-1} \\ x_2^{2e-1}-x_2^{1e-1}\end{array}\right]\left(\begin{array}{c}\Phi^{1e-1}(1-\eta) \\ +\Phi^{2e-1}\eta\end{array}\right)d\eta \\
&\quad + \int_{\eta=\varepsilon}^{1}\frac{[x_1^{2e}-x_1^{1e}, x_2^{2e}-x_2^{1e}]\eta}{l_e^2\eta^2}\left[\begin{array}{c} x_1^{2e}-x_1^{1e} \\ x_2^{2e}-x_2^{1e}\end{array}\right]\left(\begin{array}{c}\Phi^{1e}(1-\eta) \\ +\Phi^{2e}\eta\end{array}\right)d\eta \\
&= \int_{\eta=0}^{1-\varepsilon}\frac{-1}{1-\eta}\left[\Phi^{1e-1}(1-\eta)+\Phi^{2e-1}\eta\right]d\eta \\
&\quad + \int_{\eta=\varepsilon}^{1}\frac{1}{\eta}\left[\Phi^{1e}(1-\eta)+\Phi^{2e}\eta\right]d\eta \\
&= -\Phi^{1e-1} - \Phi^{2e-1}\int_{\eta=0}^{1-\varepsilon}\frac{\eta}{1-\eta}d\eta + \Phi^{1e}\int_{\eta=\varepsilon}^{1}\frac{(1-\eta)}{\eta}d\eta + \Phi^{2e} \\
&= \Phi^{2e} - \Phi^{1e-1} + \Phi^{2e-1}\int_{\eta'=1}^{\varepsilon}\frac{(1-\eta')}{\eta'}d\eta' + \Phi^{1e}\int_{\eta=\varepsilon}^{1}\frac{(1-\eta)}{\eta}d\eta \\
&= \Phi^{2e} - \Phi^{1e-1}
\end{aligned} \tag{5.27}$$

The unit normal vector is for straight boundary elements

$$\mathbf{n} = \frac{1}{l_e}\left[\begin{array}{c} -x_2^{2e}+x_2^{1e} \\ x_1^{2e}-x_1^{1e}\end{array}\right] \tag{5.28}$$

from which due to the linear approximation of this element at all collocation points, e.g., at the initial point $\xi_i = x_i^{1e}$

$$r_{,n} = r_{,i} n_i = \frac{-\left[x_1^{1e} - x_1^{2e}, x_2^{1e} - x_2^{2e}\right] \eta}{l_e \eta} \cdot \frac{1}{l_e} \begin{bmatrix} -x_2^{2e} + x_2^{1e} \\ x_1^{2e} - x_1^{1e} \end{bmatrix} = 0 \tag{5.29}$$

at the middle point $\xi_i = \frac{1}{2}(x_i^{1e} + x_i^{2e})$

$$r_{,n} = r_{,i} n_i = \frac{\left[x_1^{1e} - x_1^{2e}, x_2^{1e} - x_2^{2e}\right] \left(\frac{1}{2} - \eta\right)}{l_e \mid \frac{1}{2} - \eta \mid} \cdot \frac{1}{l_e} \begin{bmatrix} -x_2^{2e} + x_2^{1e} \\ x_1^{2e} - x_1^{1e} \end{bmatrix} = 0 \tag{5.30}$$

and also at the end point $\xi_i = x_i^{2e}$

$$r_{,n} = r_{,i} n_i = \frac{\left[x_1^{1e} - x_1^{2e}, x_2^{1e} - x_2^{2e}\right] (1 - \eta)}{l_e \mid 1 - \eta \mid} \cdot \frac{1}{l_e} \begin{bmatrix} -x_2^{2e} + x_2^{1e} \\ x_1^{2e} - x_1^{1e} \end{bmatrix} = 0 \tag{5.31}$$

follows that the normal derivative $r_{,n} = 0$ on straight elements for all types of shape functions, i.e., an integral containing $r_{,n}$ gives zero on straight boundary parts.

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