

Principles of Boundary Element Methods

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1 Introduction

1.1 Definition

Boundary integral equations are a classical tool for the analysis of boundary value problems for partial differential equations. The term “boundary element method” (BEM) denotes any method for the approximate numerical solution of these boundary integral equations. The approximate solution of the boundary value problem obtained by BEM has the distinguishing feature that it is an exact solution of the differential equation in the domain and is parametrized by a finite set of parameters living on the boundary.

1.2 Advantages

The BEM have some advantages over other numerical methods like finite element methods (FEM) or finite differences:

1. Only the boundary of the domain needs to be discretized. Especially in two dimensions where the boundary is just a curve this allows very simple data input and storage methods.
2. Exterior problems with unbounded domains but bounded boundaries are handled as easily as interior problems.
3. In some applications, the physically relevant data are given not by the solution in the interior of the domain but rather by the boundary values of the solution or its derivatives. These data can be obtained directly from the solution of boundary integral equations, whereas boundary values obtained from FEM solutions are in general not very accurate.
4. The solution in the interior of the domain is approximated with a rather high convergence rate and moreover, the same rate of convergence holds for all derivatives of any

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order of the solution in the domain. There are difficulties, however, if the solution has to be evaluated close to, but not on the boundary.

1.3 Difficult parts

Some main difficulties with BEM are the following:

1. Boundary integral equations require the explicit knowledge of a fundamental solution of the differential equation. This is available only for linear partial differential equations with constant or some specifically variable coefficients. Problems with inhomogeneities or nonlinear differential equations are in general not accessible by pure BEM. Sometimes, however, a coupling of FEM and BEM proves to be useful, see section 7.
2. For a given boundary value problem there exist different boundary integral equations and to each of them several numerical approximation methods. Thus every BEM application requires that several choices be made. To evaluate the different possibilities, one needs a lot of mathematical analysis. Although the analysis of BEM has been a field of active research in the past decade, it is by no means complete. Thus there exist no error estimates for several methods that are widely used. From a mathematical point of view, these methods, which include very popular ones for which computer codes are available, are in an experimental state, and there might exist problems of reliability.
3. The reason for the difficulty of the mathematical analysis is that boundary integral equations frequently are not ordinary Fredholm integral equations of the second kind. The classical theory of integral equations and their numerical solution concentrates on second kind integral equations with regular kernel, however. Boundary integral equations may be of the first kind, and the kernels are in general singular. If the singularities are not integrable, one has to regularize the integrals which are then defined in a distributional sense. The theoretical framework for such integral equations is the theory of pseudodifferential operators. This theory was developed 20 years ago and is now a classical part of Mathematical Analysis [16, 19], but it is still not very popular within Applied Mathematics.
4. If the boundary is not smooth but has corners and edges, then the solution of the boundary value problem has singularities at the boundary. This happens also if the boundary conditions are discontinuous, e.g. in mixed boundary value problems. BEM clearly have to treat these singularities more directly than FEM. Because the precise shape of the singularities frequently contains important information, e.g. stress intensity factors in fracture mechanics, this is a positive aspect of BEM. But besides practical problems with the numerical treatment of these singularities, non-smooth domains also present theoretical difficulties. These have so far been satisfactorily resolved only for two-dimensional problems. The analysis of BEM for three-dimensional domains with corners and edges is still in a rather incomplete stage.

1.4 Literature

The usefulness of BEM in engineering problems is documented in a literature that amounts to several thousand pages every year. I will not attempt to give an overview of the wide range of possible applications. One can find such an overview e.g. in one of the proceedings

of the annual conference on Boundary Elements [5] or in the books [4] or [8]. A review of BEM computer codes can be found in [17]. Introductions to the theory and applications of BEM can e.g. be found in the books [7, 6, 9, 29] or in [25, 24].

1.5 Contents

In the following sections I shall first describe the general structure of a BEM application and illustrate it with the simple example of the Dirichlet problem from potential theory.

Section 3 contains a general method for deriving boundary integral equations for general elliptic boundary value problems.

Section 4 describes boundary integral equations for examples from scattering theory, elasticity theory, and heat conduction.

Discretization methods and their convergence are described in section 5, and section 6 contains some remarks on the treatment of singularities in BEM. The final section 7 contains a short description of the coupling of FEM and BEM in terms of a (new) symmetric method, for which also a convergence proof is given.

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2 The structure of a BEM application

2.1 A scheme

A typical application of BEM consists of the following parts:

- Mathematical model
- Representation formula
- Boundary integral equation
- Boundary elements
- Discrete equations
- Solution of the linear system
- Interpretation

2.2 Discussion of the scheme

For the following discussion of these parts, the Dirichlet problem for Laplace's equation will be used as an illustration.

2.2.1

The **mathematical model** is a boundary value problem for a partial differential equation. An example is the Dirichlet problem

$$\Delta u = 0 \quad \text{in a domain } \Omega \subset R^n \quad (2.1)$$

$$u = g \quad \text{on the boundary } \Gamma := \partial\Omega. \quad (2.2)$$

It is convenient to have a homogenous differential equation and inhomogeneous boundary data. A necessity is to know explicitly a *fundamental solution* of the differential equation. In the example one has the fundamental solution

$$\begin{aligned} \gamma(x, y) &= -\frac{1}{2\pi} \log |x - y| & (x, y \in R^2) & \quad \text{for } n = 2 \\ \gamma(x, y) &= \frac{1}{4\pi|x-y|} & (x, y \in R^3) & \quad \text{for } n = 3. \end{aligned}$$

Fundamental solutions are known for equations with constant coefficients where they can be computed by Fourier transformation, and for some elliptic equations with analytic coefficients, e.g. the Laplace-Beltrami equation on the sphere.

2.2.2

The next step is to represent the solution of the partial differential equation in the domain by means of boundary potentials. Such **representation formulas** are well known for the classical boundary value problems of mathematical physics, e.g. Green's third identity for potential theory, Betti's formula for elasticity theory, and the Stratton-Chu formula for electrodynamics. In potential theory we have

$$u(x) = \int_{\Gamma} \partial_{n(y)} \gamma(x, y) [u(y)]_{\Gamma} do(y) - \int_{\Gamma} \gamma(x, y) [\partial_n u(y)]_{\Gamma} do(y) \quad (2.3)$$

for $x \in R^n \setminus \Gamma$.

Here ∂_n denotes the normal derivative taken with respect to the exterior normal on Γ , do is the surface measure on Γ , and u is supposed to be a harmonic function regular both in Ω and the exterior domain $R^n \setminus \bar{\Omega}$, having i.g. different boundary values on both sides of Γ , whose jump across Γ is denoted by $[\cdot]_{\Gamma}$:

$$[v(x)]_{\Gamma} := v|_{R^n \setminus \Omega}(x) - v|_{\bar{\Omega}}(x) \quad (x \in \Gamma).$$

Thus u is represented as the sum of a double layer and a simple layer potential. If considered in Ω alone, the representation formula (2.3) can take various different forms depending on assumptions on u in $R^n \setminus \Omega$. Each of these forms gives rise to a different BEM, so one has to make a choice here. Most common are the following:

1. Supposing $u|_{R^n \setminus \Omega} \equiv 0$, one obtains

$$u(x) = - \int_{\Gamma} \partial_{n(y)} \gamma(x, y) u(y) do(y) + \int_{\Gamma} \gamma(x, y) \partial_n u(y) do(y) \quad (x \in \Omega) \quad (2.4)$$

The BEM derived from this is called "method of Green's formula" or "direct method". The densities of the simple and double layer on the boundary are the normal derivative of the solution and the solution, respectively, on the boundary. Thus these layers have a "direct" interpretation in terms of the solution.

2. Supposing $[u]_\Gamma \equiv 0$, one obtains a simple layer representation

$$u(x) = \int_\Gamma \gamma(x, y)\psi(y)do(y) \quad (x \in \Omega) \quad (2.5)$$

with an unknown density ψ .

3. Supposing $[\partial_n u]_\Gamma \equiv 0$, one obtains a double layer representation

$$u(x) = \int_\Gamma v(y)\partial_n \gamma(x, y)do(y) \quad (x \in \Omega) \quad (2.6)$$

with an unknown density v .

Sometimes also a suitable linear combination of simple and double layer potentials will be useful.

2.2.3

The representation formulas give the solution u in the interior of the domain Ω . If one takes boundary values in the representation formula, one obtains **boundary integral equations**. Here one can i.g. also choose between different possibilities. In our example, these are as follows:

In the “indirect methods” 2. and 3. above, one will insert the representation formulas (2.5) or (2.6) into the boundary condition (2.2). One has to take into account the well-known “jump relations” which arise from the fact that the gradient of γ has a strong singularity and is not integrable on Γ .

The simple layer potential $\int_\Gamma \gamma(x, y)\psi(y)do(y)$ is continuous across Γ , thus (2.5) yields the integral equation

$$V\psi(x) := \int_\Gamma \gamma(x, y)\psi(y)do(y) = g(x) \quad \text{on } \Gamma. \quad (2.7)$$

This is a Fredholm integral equation of the first kind with a weakly singular kernel.

For the double layer potential (2.6) there holds

$$u(x) = -\frac{1}{2}v(x) + \int_\Gamma v(y)\partial_n \gamma(x, y)do(y) =: -\frac{1}{2}v(x) + Kv(x) \quad \text{on } \Gamma.$$

Here the integral on Γ has to be understood in the principal value sense, but it turns out that for smooth boundaries the kernel is continuous. Therefore, the integral equation arising from (2.6),

$$\left(-\frac{1}{2} + K\right)v = g \quad (2.8)$$

is a Fredholm integral equation of the second kind. (In fact it was, together with its adjoint, the starting point of the whole theory of integral equations as developed by C. Neumann, I. Fredholm, and D. Hilbert in 1870–1910.) Note, however, that in the presence of edges and corners, (2.8) has a strongly singular kernel and is no longer a Fredholm integral equation.

The representation formula (2.4) admits two ways of approaching the boundary:

If one considers just $u|_\Gamma$, one obtains

$$u = \frac{1}{2}u - Ku + V(\partial_n u) \quad \text{on } \Gamma.$$

But one can also take $\partial_n u|_\Gamma$ and observe the jump relations:

The normal derivative of the double layer potential is continuous across Γ , if the boundary and the density are smooth, and for $x \in \Gamma$

$$\partial_{n(x)} \int_\Gamma \gamma(x, y) \psi(y) d\sigma(y) = \frac{1}{2} \psi(x) + \int_\Gamma \partial_{n(x)} \gamma(x, y) \psi(y) d\sigma(y) =: \frac{1}{2} \psi(x) + K' \psi(x).$$

Here, K' is the adjoint operator of K above. Thus one obtains

$$\partial_n u = Du + \frac{1}{2} \partial_n u + K' \partial_n u \quad \text{on } \Gamma,$$

where

$$Du(x) := -\partial_{n(x)} \int_\Gamma \partial_{n(y)} \gamma(x, y) u(y) d\sigma(y) \quad \text{on } \Gamma$$

is the normal derivative of the double layer potential. The “integral operator” D has a hypersingular kernel.

We have now found the two relations on Γ

$$\left(\frac{1}{2} + K\right) u = V(\partial_n u) \tag{2.9}$$

$$Du = \left(\frac{1}{2} - K'\right)(\partial_n u). \tag{2.10}$$

Both relations together express simply the fact that the two functions $u|_\Gamma$ and $\partial_n u|_\Gamma$, the “Cauchy data” of u , are derived from a function u that satisfies the differential equation (2.1) in Ω . A method to derive such relations for more general elliptic equations is described in section 3.

Now either one of the equations (2.9), (2.10) together with the boundary condition (2.2) can be used to determine $\partial_n u|_\Gamma$ and thus, with the representation formula (2.4), to solve the Dirichlet problem. From (2.9) one obtains

$$V(\partial_n u) = \left(\frac{1}{2} + K\right) g, \tag{2.11}$$

an integral equation of the first kind with the same kernel as in (2.7). From (2.10) one obtains

$$\left(\frac{1}{2} - K'\right)(\partial_n u) = Dg, \tag{2.12}$$

an integral equation of the second kind with the adjoint kernel of (2.8).

Any one of the four integral equations (2.7), (2.8), (2.11), or (2.12) can equally well be used for the numerical solution of the Dirichlet problem.

It is an easy exercise to write down the corresponding integral equations for the Neumann problem where $\partial_n u$ is given on Γ , and also for the Robin problem where a linear combination of u and $\partial_n u$ is given on Γ , and even for mixed boundary value problems where on different parts of Γ different boundary conditions are satisfied. In each case, only the four operators V , D , K , and K' appearing in (2.9) are involved.

Two final remarks on the variety of boundary integral equations:

- In representation formulas like (2.4), γ can be *any* fundamental solution. Thus if Γ has a simple geometry that allows to find a fundamental solution which vanishes on parts of Γ , the integrals may simplify considerably. For instance, if γ is the *Green function* of Ω (which is known e.g. for balls or half spaces), then $\gamma(x, y) \equiv 0$ for $y \in \Gamma$, hence the direct method in this case coincides with the method of double layer representation.

- If Γ is an open surface which does not separate R^n into two disjoint domains, one has no choice but has to work with (2.3) directly. This rules out all integral equations of the second kind and leaves for the Dirichlet problem the integral equation of the first kind with the operator V and for the Neumann problem the integral equation of the first kind with the operator D (see [22, 23]). Such situations occur in crack problems or in screen scattering problems.

2.2.4

For the numerical solution of the boundary integral equations one needs functions on the boundary that depend on a finite set of parameters, i.e. a finite dimensional function space on Γ . One can choose globally defined functions like spherical harmonics or polynomials, but for the proper BEM these trial functions are finite element functions on the boundary, the **boundary elements**. One assumes that Γ can be decomposed into a finite number of subsets each of which has a regular parameter representation by some parameter domain in R^{n-1} . Then one chooses regular partitions of the parameter domains and corresponding finite element functions, e.g. a $S_h^{k,m}$ family in the sense of Babuška and Aziz [2]. Using the parameter representations, one transfers these functions to the boundary. If the parameter representation is not explicitly known but the boundary itself has also to be approximated, one uses the approximate coordinate representations. The order of the polynomial basis functions in these approximations of course has to be chosen in correspondence to the order of the Sobolev spaces in which the solution of the boundary integral equation is looked for and also to the kind of discretization scheme that is chosen. Here it is useful to have asymptotic error estimates in order to make the right choice. This will be discussed in section 5.

If, in the presence of corners and edges, one knows which singularities appear in the solution, then one can build them into the space of trial functions as in FEM, either by using a suitable nonuniform partition or by augmenting the space of trial functions by some singular functions, i.e. using singular boundary elements.

2.2.5

The finitely many parameters determining the approximate solution are computed from finitely many linear equations. There are three main methods to generate this system of **discrete equations**: Collocation, Galerkin's method, and the least squares method.

If we have a boundary integral equation

$$Au = f \quad \text{on } \Gamma \tag{2.13}$$

and we seek an approximate solution

$$u_h(x) = \sum_{j=1}^N \gamma_j \mu_h^j(x) \tag{2.14}$$

with basis functions $\{\mu_h^j | j = 1, \dots, N\}$, then these methods for obtaining a linear system for the unknown coefficients $\{\gamma_j | j = 1, \dots, N\}$ are the following:

1. For the **collocation method** one chooses a suitable set $\{x_j | j = 1, \dots, N\} \subset \Gamma$ of collocation points and requires that the equation (2.13) is satisfied in these points. This

gives the system

$$\sum_{j=1}^N (A\mu_h^j)(x_k) \gamma_j = f(x_k) \quad (k = 1, \dots, N). \quad (2.15)$$

2. For the **Galerkin method** one multiplies (2.13) for u_h with test functions from a finite dimensional function space, integrates over Γ and equates the integrals. In the proper Galerkin method, test and trial functions are the same, hence this gives the system

$$\sum_{j=1}^N (\mu_h^k, A\mu_h^j) \gamma_j = (\mu_h^k, f) \quad (k = 1, \dots, N) \quad (2.16)$$

where the brackets (\cdot, \cdot) denote the $L^2(\Gamma)$ inner product, i.e.

$$(f, g) := \int_{\Gamma} \overline{f(x)} g(x) d\sigma(x).$$

3. In the **least squares method** one minimizes $Au_h - f$ in the $L^2(\Gamma)$ norm. The resulting equations are those for a Galerkin (-Petrov) method with $\{A\mu_h^k | k = 1, \dots, N\}$ as a basis for the test functions:

$$\sum_{j=1}^N (A\mu_h^k, A\mu_h^j) \gamma_j = (A\mu_h^k, f) \quad (k = 1, \dots, N). \quad (2.17)$$

The coefficient matrix in these methods has to be computed using numerical integration. Here the collocation method is the simplest one because it involves only one integral whereas Galerkin's method needs two and the least squares method three integrations. This may be the reason why collocation is most frequently used in the applications. It has also some drawbacks, however:

The coefficient matrix in the least squares method is always symmetric and, if (2.13) is uniquely solvable, even positive definite. A careful choice of the boundary integral operator can yield a positive definite selfadjoint operator A or at least a positive definite principal part. For instance, the first kind operators V and D from potential theory are selfadjoint and, in R^3 , positive definite. In the Galerkin method, then the coefficient matrix will also be positive selfadjoint. The coefficient matrix for the collocation method is never symmetric.

Convergence proofs and asymptotic error estimates are available

- for least squares methods under the general condition of *ellipticity* for the operator A ,
- for Galerkin methods under the somewhat stronger condition of *strong ellipticity* for the operator A ,
- for collocation methods only in 2 dimensions for *strongly elliptic* operators A and *smooth* domains (apart from some very special recent results [14, 10]) and in higher dimensions only for Fredholm integral equations of the second kind (for which, on the other hand, there exist also a lot of other numerical methods, e.g. the well-known Nyström method, see [3]).

The integrations in the computation of the coefficient matrix have to be done numerically. This requires another choice which, however, depends very much on the specific problem at hand. A common problem is the evaluation of singular integrals, and the usual recipe is to split off the main singularities and try to evaluate the corresponding integrals analytically, while the less singular remainder is computed with the use of a suitable quadrature formula. The order of this quadrature formula should be chosen in accordance with the asymptotic order of convergence for the chosen discretization method.

2.2.6

In comparison with the computation of the matrix elements, the **solution of the linear system** of equations is usually relatively uncomplicated. Note that the matrices are not sparse, so the linear systems solvers developed for FEM are not useful in BEM. In some specific examples, multigrid methods have been analyzed and successfully applied [20]. In general, standard linear systems solvers should suffice, if one exploits symmetry and positive definiteness where available.

2.2.7

The postprocessing, i.e. the **interpretation** of the computed data, is the last step in BEM. The interpolation formula (2.14) gives the solution of the boundary integral equation. In the direct method, this may be already the desired information about the boundary values of the solution of the boundary value problem. If these Cauchy data have to be determined from an indirect method, then another integral operator with singular kernel has to be applied.

If the solution of the boundary value problem in the interior of Ω is desired, then the representation formula which was used to derive the boundary integral equation gives the answer. Here only a regular integral has to be evaluated, because the kernel defined by the fundamental solution is, for elliptic and parabolic problems, singular only if the points of integration and of evaluation coincide. Thus this integration has a smoothing effect and suppresses high-frequency errors. In addition, the representation formula may be differentiated and used to compute the derivatives of the solution in the domain.

3 Derivation of boundary integral equations

Let P be a linear elliptic differential operator of order $2m$ with smooth coefficients on R^n . Assume that a fundamental solution G of P is known, that is

- G is a two-sided inverse of P on functions (or distributions) that vanish outside a compact set,

$$GPu = u = PGu \quad \text{for all compactly supported } u.$$

- G is an integral operator

$$Gf(x) = \int_{R^n} G(x, y)f(y)dy$$

with a kernel $G(x, y)$ that is weakly singular for $x = y$.

In this section we describe the method of the *Calderón projector* to obtain boundary integral equations for boundary value problems of the following form

$$Pu = f \quad \text{in } \Omega \subset R^n \quad (3.1)$$

$$R\gamma u = g \quad \text{on } \Gamma = \partial\Omega. \quad (3.2)$$

Here

$$\gamma u := (\gamma_0 u, \dots, \gamma_{2m-1} u)^\top := (u, \partial_n u, \partial_n^2 u, \dots, \partial_n^{2m-1} u)^\top_\Gamma$$

are the ‘‘Cauchy data’’ of u , and R is a $(m \times 2m)$ matrix of differential operators on Γ . Thus the boundary data g are a m -component vector function.

3.1 Second Green formula

Let $\Omega_1 := \Omega$ and $\Omega_2 := R^n \setminus \bar{\Omega}$. For a function u on R^n with $u|_{\Omega_j} = u_j$ we write $[\gamma_k u]_\Gamma = \gamma_k u_2 - \gamma_k u_1$ if the Cauchy data $\gamma_k u_j = \partial_n^k (u_j)_\Gamma$ exist. Here one assumes that in a neighborhood of Γ there exists a unit vector field \vec{n} that is the normal field exterior to Ω_1 on Γ , and the higher normal derivatives are defined using this vector field.

Introducing suitable coordinates, one can see that P has a decomposition

$$P = \sum_{j=0}^{2m} P_j \partial_n^j \quad (3.3)$$

Here P_j are differential operators of order $2m - j$ whose restrictions to Γ contain only tangential derivatives.

For example, the Laplace operator $P = -\Delta$ is decomposed as

$$-\Delta = -\Delta_0 - 2H\partial_n - \partial_n^2,$$

where $H = \frac{1}{2} \operatorname{div} \vec{n}$ is the mean curvature of Γ and Δ_0 is the Laplace-Beltrami operator on Γ .

Now we choose a piecewise smooth function u as above such that $u_j \in C^\infty(\bar{\Omega}_j)$. We apply P to u in the distributional sense. One has

$$Pu = f \quad \text{in } R^n \setminus \Gamma \quad \text{with } f \in C^\infty(R^n \setminus \Gamma).$$

Therefore the distribution $Pu - f$ vanishes on $R^n \setminus \Gamma$. It consists of multiple layers of the form $v \otimes \partial_n^k \delta_\Gamma$ defined by

$$\langle v \otimes \partial_n^k \delta_\Gamma, \varphi \rangle := \int_\Gamma v \partial_n^k \varphi \, d\sigma \quad \text{for } \varphi \in C^\infty(\Gamma).$$

Here ∂_n^k is the formal transpose of the differential operator ∂_n^k , hence

$$\partial_n^k = (-1)^k \partial_n^k + \text{lower order terms.}$$

The result can be found in detail in [16]. It is the distributional version of the **second Green formula**:

$$Pu = f + \sum_{k=0}^{2m-1} \sum_{l=0}^{2m-1-k} P_{k+l+1} [\gamma_l u]_\Gamma \otimes \partial_n^k \delta_\Gamma. \quad (3.4)$$

If we specialize to $u_2 \equiv 0$, $u_1 = v$, and apply (3.4) to a test function φ , we obtain the more familiar equivalent formula

$$\int_{\Omega} (v^{\top} P' \varphi - (Pv)^{\top} \varphi) dx = \sum_{k=0}^{2m-1} \sum_{l=0}^{2m-1-k} \int_{\Gamma} (P_{k+l+1} \gamma_l v)^{\top} (\partial_n^k \varphi) do. \quad (3.5)$$

Here the superscript \top denotes transposition, because we admit throughout matrix-valued operators P and vector-valued functions u , v etc., and P' is the formal transpose of P :

$$\langle Pu, \varphi \rangle = \langle u, P' \varphi \rangle = \int_{\Omega} v^{\top} P' \varphi dx \quad \text{for } \varphi \in C_0^{\infty}(R^n).$$

The formulas (3.4) and (3.5) are equivalent, but it is the distributional form (3.4) that is the starting point for the representation formula and boundary integral equations. In the example of the Laplace operator, (3.4) is

$$-\Delta u = f - [\gamma_1 u]_{\Gamma} \otimes \delta_{\Gamma} + [\gamma_0 u]_{\Gamma} \otimes \partial_n' \delta_{\Gamma}.$$

In the case of a second order operator (we abbreviate $\partial_j := \partial/\partial x_j$)

$$P = - \sum_{j,k=1}^n \partial_j a_{jk} \partial_k + \sum_{j=1}^n b_j \partial_j + c \quad (3.6)$$

one has a *first Green formula*

$$\int_{\Omega} (Pu)^{\top} v dx = \Phi_{\Omega}(u, v) - \int_{\Gamma} (\partial_{\nu} u)^{\top} v do \quad (3.7)$$

with

$$\Phi_{\Omega}(u, v) := \int_{\Omega} \left(\sum_{j,k=1}^n (a_{jk} \partial_k u)^{\top} \partial_j v + \sum_{j=1}^n (b_j \partial_j u)^{\top} v + (cu)^{\top} v \right) dx$$

and the conormal derivative

$$\partial_{\nu} u := \sum_{j,k=1}^n n_j a_{jk} \partial_k u.$$

Here again the coefficients a_{jk} , b_j , and c may be matrix-valued. Then, for example, the equations of linear elasticity can be treated in this way, where then ∂_{ν} is the traction operator.

From (3.7) follows the second Green formula

$$\int_{\Omega} (u^{\top} P' v - (Pu)^{\top} v) dx = \int_{\Gamma} ((\partial_{\tilde{\nu}} u)^{\top} v - u^{\top} \tilde{\partial}_{\nu} v) do \quad (3.8)$$

with

$$\partial_{\tilde{\nu}} u := \partial_{\nu} u - \sum_{j=1}^n n_j b_j u ; \quad \tilde{\partial}_{\nu} v := \sum_{j,k=1}^n n_j a_{jk}^{\top} \partial_k v.$$

In the case of elasticity, (3.8) is Somigliana's identity.

3.2 Representation formula

From the second Green formula in the distributional form, one obtains the representation formula simply by applying the inverse G of the differential operator P . Thus if u is piecewise smooth as above and vanishes outside a compact set, then there holds the **representation formula**

$$u = Gf + \sum_{k=0}^{2m-1} \sum_{l=0}^{2m-1-k} K_k(P_{k+l+1}[\gamma l u]_\Gamma) \quad (3.9)$$

with the multiple layer potential operators K_k defined by

$$K_k \varphi := G(\varphi \otimes \partial_n^k \delta_\Gamma) \quad \text{for } \varphi \in C^\infty(\Gamma).$$

For $x \notin \Gamma$ this means

$$K_k \varphi(x) = \int_\Gamma \partial_{n(y)}^k G(x, y) \varphi(y) do(y).$$

In the example of the Laplace operator, this is just (2.3) if $f = 0$.

For the second order operator (3.6) one obtains from (3.8)

$$u(x) = Gf(x) + \int_\Gamma \left((\widetilde{\partial_{\nu(y)}} G(x, y)^\top)^\top [u(y)]_\Gamma - G(x, y) \partial_{\bar{\nu}} u(y) \right) do(y). \quad (3.10)$$

For this case of a second order operator, one can allow corners and edges on Γ , even an arbitrary Lipschitz domain [12], whereas for the general case (3.9) the boundary has to be smooth. For the bi-Laplace operator $P = \Delta^2$ in R^2 also representation formulas are known in domains with corners, and they contain additional terms coming from the corners.

3.3 Calderón projector

Taking traces on Γ in (3.9), we find an identity for the Cauchy data γu_1 , if we set $u_2 \equiv 0$:

$$\gamma u_1 = \gamma Gf_1 + C(\gamma u_1) \quad (3.11)$$

with

$$C(\gamma u_1) := - \sum_{k=0}^{2m-1} \sum_{l=0}^{2m-1-k} \gamma(K_k(P_{k+l+1} \gamma l u_1))|_\Omega.$$

The operator C is a $(2m \times 2m)$ matrix of pseudodifferential operators on the boundary. In the case of the Laplace operator, it is in the notation of section 2.2.3

$$C = \begin{pmatrix} \frac{1}{2} - K & V \\ D & \frac{1}{2} + K' \end{pmatrix}.$$

One can show that C is a projection operator: $CCv = Cv$ for all v . The complementary projector $1 - C$ is the Calderón projector for the complementary domain.

As a pseudodifferential operator, C has well-known continuity properties in *Sobolev spaces* on Γ . I will not give the definition of a Sobolev space $H^s(\Gamma)$ for arbitrary $s \in R$ here, but recall that for integer s , $H^s(\Gamma)$ is the space of all functions on Γ with finite norm

$$\|v\|_{H^s(\Gamma)} := \left(\sum_{k \leq s} \int_\Gamma |D^k v|^2 do \right)^{\frac{1}{2}},$$

where $D^k v$ is the vector of all k -th order tangential derivatives of v . The dual space of the Hilbert space $H^s(\Gamma)$ in the natural $L^2(\Gamma)$ duality is $H^{-s}(\Gamma)$.

It is known that $C = (C_{jk})_{j,k=0}^{2m-1}$ where C_{jk} is a pseudodifferential operator of order $j - k$. This means that C_{jk} maps $H^s(\Gamma)$ into $H^{s-j+k}(\Gamma)$ continuously for any s . Thus the boundary “integral” operators C_{jk} have orders ranging from $-2m + 1$ to $2m - 1$. Only those with nonpositive order are decent integral operators, where order zero operators can contain strongly singular kernels and thus integrals existing only in the principal value sense. The operators with positive order have hypersingular kernels like the operator D which has order $+1$.

But all of these highly nonclassical “integral” operators do appear in applications and they can be used in BEM.

3.4 Boundary integral equations

The relation (3.11) is a faithful translation of the differential equation (3.1) to the boundary: A $2m$ -vector function v satisfies $v = \gamma Gf + Cv$ if and only if v equals γu with some solution u of (3.1). Thus (3.11) together with the boundary condition (3.2) is an equivalent formulation of the boundary value problem by means of functions on the boundary. It is a system of $3m$ equations for the $2m$ unknowns $\gamma_0 u, \dots, \gamma_{2m-1} u$.

There are several ways of making a quadratic system out of this rectangular one. One way would be to treat the $3m \times 2m$ system directly by least squares methods. This would lead to a $2m \times 2m$ system. The ellipticity of this system, i.e. the invertibility of its symbol as a pseudodifferential operator, is equivalent to the *Shapiro-Lopatinski* ellipticity condition for the boundary value problem (3.1)–(3.2).

The usual procedure, however, is to use the boundary condition to eliminate m unknowns and find a $m \times m$ system for the remaining m unknown functions. This procedure uses a second set S of m boundary operators with the property that the $(2m \times 2m)$ matrix

$$M := \begin{pmatrix} R \\ S \end{pmatrix}$$

of tangential differential operators on Γ has an inverse

$$M^{-1} = (\check{M}_{jk})_{j,k=0}^{2m-1}$$

which also consist of differential operators. Then the m -component vector function

$$v = S\gamma u$$

contains the unknown Cauchy data, and the Cauchy data γu are known if v is known and g is given by (3.2):

$$\gamma u = M^{-1} \begin{pmatrix} g \\ v \end{pmatrix}. \quad (3.12)$$

In this case, the solution of the boundary value problem in Ω is given by the representation formula (3.9):

$$u = Gf - \mathcal{K}\mathcal{P}M^{-1} \begin{pmatrix} g \\ v \end{pmatrix}, \quad (3.13)$$

where we introduced the vector of integral operators

$$\mathcal{K} := (K_0, \dots, K_{2m-1})$$

and the matrix of differential operators

$$\mathcal{P} := (P_{k+l+1})_{k,l=0}^{2m-1} \quad \text{with } P_j := 0 \text{ for } j > 2m.$$

Taking the “modified Cauchy data” $M\gamma u$ in (3.13), we have the $2m \times 2m$ system

$$\begin{pmatrix} g \\ v \end{pmatrix} = M\gamma Gf - M\gamma \mathcal{K} \mathcal{P} M^{-1} \begin{pmatrix} g \\ v \end{pmatrix},$$

which can of course be also written in the form

$$\begin{pmatrix} g \\ v \end{pmatrix} = M\gamma Gf + \tilde{C} \begin{pmatrix} g \\ v \end{pmatrix}$$

with the “modified Calderón projector”

$$\tilde{C} := MCM^{-1}.$$

We write this system finally in the form

$$g = R\gamma Gf + RCM^{-1} \begin{pmatrix} g \\ v \end{pmatrix}$$

$$v = S\gamma Gf + SCM^{-1} \begin{pmatrix} g \\ v \end{pmatrix}$$

or

$$Av = g - RCM^{-1} \begin{pmatrix} g \\ 0 \end{pmatrix} - R\gamma Gf \tag{3.14}$$

$$(1 - B)v = SCM^{-1} \begin{pmatrix} g \\ 0 \end{pmatrix} + S\gamma Gf \tag{3.15}$$

where

$$Av = RCM^{-1} \begin{pmatrix} 0 \\ v \end{pmatrix} = -R\gamma \mathcal{K} \mathcal{P} M^{-1} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

and

$$Bv = SCM^{-1} \begin{pmatrix} 0 \\ v \end{pmatrix} = -S\gamma \mathcal{K} \mathcal{P} M^{-1} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

are the upper resp. lower right hand “ $(m \times m)$ corner” of the $(2m \times 2m)$ matrix \tilde{C} .

Now either of the two sets of m equations (3.14) or (3.15) is equivalent to the original boundary value problem under suitable conditions which can be found in [15]:

Theorem 3.1 *Let $f \in H^{s-m}(\Omega) \cap L^2(\Omega)$ and $g \in \prod_{j=0}^{m-1} H^{m-\mu_j-\frac{1}{2}+s}(\Gamma)$ be given where s is an arbitrary real number and μ_j are the orders of the rows of the boundary differential operator R . Then every solution $u \in H^{m+s}(\Omega)$ of the boundary value problem (3.1), (3.2) defines by (3.12) a solution of (3.14) and (3.15). Conversely, every solution $v \in \prod_{j=0}^{m-1} H^{-m+\mu_j+\frac{1}{2}+s}(\Gamma)$ of one of the boundary integral equations, (3.14) or (3.15), defines by the representation formula (3.13) a solution $u \in H^{m+s}(\Omega)$ of the boundary value problem.*

Under some conditions on the strong ellipticity of the boundary value problem (3.1), (3.2), one can show [15] that the operator A in (3.14) is also strongly elliptic. These conditions are satisfied for the usual boundary value problems for the Laplace and Helmholtz equations and for the equations of linear elasticity theory, whether formulated as the second order system of the Navier equations or, through the Airy stress function, as the fourth order scalar biharmonic equation. They are not satisfied for some boundary value problems from electrodynamics.

Strong ellipticity of the operator A implies

1. Fredholm's alternative holds for the boundary integral equation (3.14), i.e. the homogeneous equation has a finite number l of linear independent solutions and there exist l linear independent solvability conditions. Thus, by adding a finite number of rows and columns, the system can be converted into a uniquely solvable one of the form

$$Av + \sum_{j=1}^l \omega_j v_j^0 = \psi \tag{3.16}$$

$$\int_{\Gamma} \lambda_k^{\top} v \, do = \beta_k \quad (k = 1, \dots, l),$$

where ψ is the right hand side in (3.14), v_j^0 and λ_k are certain given functions related to the cokernel and kernel of A , β_k are arbitrarily given numbers, e.g. zero, and ω_j are numbers that have to be determined from the boundary integral equation. The enlarged system is also equivalent to the boundary value problem. Such an addition of rows and columns is necessary in elasticity theory to eliminate the rigid body motions, and also sometimes in potential and scattering theory.

2. Every *Galerkin procedure* for (3.16) is stable and converges [21]. The asymptotic order of convergence is quasioptimal, i.e. it is the same order as for the best possible approximation of the solution by means of the boundary element trial functions.
3. In the case of smooth 2-dimensional domains, also *collocation methods* are stable and converge quasioptimally.

The same result of strong ellipticity for the first kind operator A in (3.14) holds for second order systems even on arbitrary Lipschitz domains Ω , i.e. in the presence of corners and edges [11].

For the second kind operator $1 - B$ in (3.15) strong ellipticity is known in special cases, mainly on smooth domains, but in some cases on non-smooth domains it fails.

4 Examples of boundary integral equations

4.1 Acoustic scattering

We consider a *screen* $\Gamma \subset R^3$, i.e. an open surface. The total field $u = u_{\text{in}} + u_s$ as well as the incident (plane or spherical) wave u_{in} and the scattered field u_s satisfy the Helmholtz equation. If u satisfies homogeneous Neumann conditions on Γ , we have for the scattered field u_s the Neumann problem

$$(\Delta + k^2) u_s = 0 \quad \text{in } R^n \setminus \Gamma \tag{4.1}$$

$$\partial_n u_s = g \quad \text{on } \Gamma \tag{4.2}$$

with $g = -\partial_n u_{\text{in}}$.

The direct method as well as the method of double layer potential representation (they coincide due to $[\partial_n u_s]_\Gamma = 0$) both lead to the first kind integral equation

$$Dv = -g \quad (4.3)$$

with the hypersingular integral operator D introduced in section 2. Here, of course, the fundamental solution γ is the outgoing one for the Helmholtz equation:

$$\gamma(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}.$$

The solution v has a direct interpretation as

$$v = [u_s]_\Gamma.$$

The operator D is (for real wave numbers k) a positive selfadjoint operator on $L^2(\Gamma)$ with domain of definition $\tilde{H}^{1/2}(\Gamma)$ which is the dual space of $H^{-1/2}(\Gamma)$, and D maps $\tilde{H}^{1/2}(\Gamma)$ bijectively onto $H^{-1/2}(\Gamma)$. Thus every Galerkin method for the equation (4.3) is convergent.

One has to choose the boundary elements in such a way that the solution $v \in \tilde{H}^{1/2}(\Gamma)$ is approximated as good as possible. In general the solution will have unbounded derivatives near the screen edge. This allows an asymptotic order of convergence of at most $\mathcal{O}(h)$ for regular finite elements. There exist methods using singular boundary elements (see section 6) that allow an order of $\mathcal{O}(h^3)$ [22].

4.2 Elasticity theory

For the system of partial differential equations

$$\mu\Delta\vec{u} + (\lambda + \mu) \text{grad div } \vec{u} = 0 \quad \text{in } \Omega \quad (4.4)$$

one has the representation formula

$$\vec{u}(x) = \int_\Gamma (\mathbf{F}(x, y)\vec{t}(y) - \mathbf{T}(y, x)\vec{\varphi}(y)) do(y) \quad \text{for } x \in \Omega, \quad (4.5)$$

where

$$\vec{\varphi} = \vec{u}|_\Gamma ; \vec{t} = T(\vec{u})$$

with the traction operator

$$T(\vec{u}) := \lambda \vec{n}(x) \text{div } \vec{u} + 2\mu \partial_n \vec{u} + \mu \vec{n} \wedge \text{curl } \vec{u}.$$

\mathbf{F} is the fundamental solution of (4.4)

$$\mathbf{F}(x, y) = \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)} \left(\gamma(x, y) + \frac{\lambda + \mu}{\lambda + 3\mu} \frac{(x-y)(x-y)^\top}{|x-y|^n} \right) \quad (n = 2, 3)$$

with γ as in section 2.2.1, and

$$\mathbf{T}(y, x) = (T_{(y)}(\mathbf{F}(y, x)))^\top.$$

Using the jump relations for elastic potentials, one obtains on Γ [28]

$$\vec{u}(x) = \frac{1}{2}\vec{\varphi}(x) - \int_{\Gamma} (\mathbf{T}(y, x)\vec{\varphi}(y) + \mathbf{F}(x, y)\vec{t}(y))do(y) \quad (4.6)$$

$$T(\vec{u})(x) = -T_{(x)}\left(\int_{\Gamma} \mathbf{T}(y, x)\vec{\varphi}(y)do(y)\right) + \frac{1}{2}\vec{t}(x) + \int_{\Gamma} \mathbf{T}(x, y)^{\top}\vec{t}(y)do(y). \quad (4.7)$$

Here the kernels on the right hand side are strongly singular, and the integrals have to be understood in the Cauchy principal value sense.

The equations (4.6), (4.7) immediately give boundary integral equations for the two boundary value problems

$$T(\vec{u}) = \vec{t} \quad \text{given on } \Gamma \quad (4.8)$$

or

$$\vec{u} = \vec{\varphi} \quad \text{given on } \Gamma. \quad (4.9)$$

Let us consider the first one (4.8). Here the unknown is $\vec{u}|_{\Gamma} = \vec{\varphi}$.

The equation (4.6) gives for $n = 3$ [28]

$$\begin{aligned} \frac{1}{2}\vec{\varphi}(x) + \int_{\Gamma} \mathbf{T}(y, x)\vec{\varphi}(y)do(y) + \mathbf{M}(x)\vec{\omega} &= \int_{\Gamma} \mathbf{F}(x, y)\vec{t}(y)do(y) \\ \int_{\Gamma} \mathbf{M}(y)^{\top}\vec{\varphi}(y)do(y) &= 0 \end{aligned} \quad (4.10)$$

with the matrix

$$\mathbf{M}(x) = \begin{pmatrix} 1 & 0 & 0 & -x_2 & 0 & x_3 \\ 0 & 1 & 0 & x_1 & -x_3 & 0 \\ 0 & 0 & 1 & 0 & x_2 & -x_1 \end{pmatrix}.$$

Here the extra 6 unknowns $\vec{\omega}$ and the extra 6 side conditions are added in order to remove the rigid body motions, thus making the system (4.10) uniquely solvable.

The integral operator in (4.10) is strongly singular. Thus (4.10) is *not* a Fredholm integral equation of the second kind. For smooth domains, however, it has been shown that the integral operator in (4.10) is a strongly elliptic pseudodifferential operator of order zero. Therefore any Galerkin method for (4.10) is convergent.

If we take the equation (4.7), we obtain the first kind equation

$$\begin{aligned} T_{(x)}\left(\int_{\Gamma} \mathbf{T}(y, x)\vec{\varphi}(y)do(y)\right) + \mathbf{M}(x)\vec{\omega} &= \frac{1}{2}\vec{t}(x) - \int_{\Gamma} \mathbf{T}(x, y)^{\top}\vec{t}(y)do(y) \\ \int_{\Gamma} \mathbf{M}(y)^{\top}\vec{\varphi}(y)do(y) &= 0. \end{aligned} \quad (4.11)$$

Here the kernel is hypersingular. But one can show that even for any Lipschitz domain the operator defines a positive definite hermitian bilinear form, and hence any Galerkin method converges for (4.11) even if the domain has corners and edges.

BEM for three-dimensional elasticity are of course very important in engineering applications. Frequently the equations are discretized using a *collocation method* (BETSY, BEASY and other programs). It is interesting to note, however, that all attempts to prove convergence for these methods have been, to the best of the author's knowledge, until now unsuccessful. This gives a theoretical argument in favor of Galerkin methods.

4.3 Heat conduction

Stationary heat conduction processes are governed by Laplace's equation which was treated above. Therefore here we concentrate on transient processes. The heat equation is

$$Lu := \partial_t u - \Delta u = 0 \quad \text{in } Q = \Omega \times (0, T) \subset R^{n+1}. \quad (4.12)$$

There exist several BEM for initial-boundary value problems for parabolic equations:

- One can choose a time-discretization scheme that requires at each time-step the solution of an elliptic boundary value problem in the space domain Ω .
- Also the application of Laplace transformation leads to a (parameter-dependent) elliptic boundary value problem. The elliptic boundary value problems are then solved by one of the BEM described above.
- With the help of a time-dependent fundamental solution one can perform BEM directly on the boundary of the space-time cylinder Q .

The latter method is described here along the lines of the general scheme described in section 3.

A fundamental solution of (4.12) is given by

$$G(x, t, y, s) = \begin{cases} (4\pi(s-t))^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4(s-t)}} & \text{for } s > t \\ 0 & \text{for } s < t. \end{cases}$$

We apply L in the distributional sense to a function u which is smooth in \overline{Q} and in $R^{n+1} \setminus Q$, satisfies $Lu = f$ in $R^{n+1} \setminus \partial Q$, and has jumps across

$$\partial Q = \overline{\Omega_0} \cup \overline{\Omega_T} \cup (\Gamma \times [0, T]) \quad (\Omega_t := \Omega \times \{t\}).$$

The result is the distributional version of the second Green formula:

$$Lu(x, t) = f(x, t) - u(x, T)\delta(t-T) + u(x, 0)\delta(t) + [u(x, t)]_\Gamma \otimes \partial'_n \delta_\Gamma(x, t) - [\partial_n u(x, t)]_\Gamma \otimes \delta_\Gamma(x, t). \quad (4.13)$$

Convolution with G gives the representation formula for $(x, t) \notin \partial Q$:

$$\begin{aligned} u(x, t) &= \int_Q G(x, t, y, s) f(y, s) dy ds + \int_\Omega (G(x, t, y, 0)u(x, 0) - G(x, t, y, T)u(x, T)) dy \\ &\quad + \int_0^T \int_\Gamma (\partial_{n(y)} G(x, t, y, s)[u(y, s)]_\Gamma - G(x, t, y, s)[\partial_n u(y, s)]_\Gamma) do(y) ds. \end{aligned}$$

Taking into account that $G = 0$ for $s < t$, this is equivalent to

$$\begin{aligned} u(x, t) &= \int_0^t \int_\Omega G(x, t, y, s) f(y, s) dy ds + \int_\Omega G(x, t, y, 0)u(x, 0) dx \\ &\quad + \int_0^t \int_\Gamma (\partial_{n(y)} G(x, t, y, s)[u(y, s)]_\Gamma - G(x, t, y, s)[\partial_n u(y, s)]_\Gamma) do(y) ds. \end{aligned} \quad (4.14)$$

If we consider the initial value problem

$$u(x, 0) = u_0(x) \quad (4.15)$$

for the equation (4.12) then on the right hand side of (4.14) the first term is absent and the second is known. (For general initial values u_0 , it has to be computed by a domain integral, which, due to the smoothing nature of the kernel $G(x, t, y, 0)$, may be discretized relatively crudely.) The remaining double and simple layer potentials resemble those for the Laplace equation. Thus one may take traces of u and $\partial_n u$ on Γ and obtain, as in section 2.2.3, two boundary integral equations for both the Dirichlet and the Neumann boundary conditions. We leave this as an exercise.

In the case of smooth Ω , the respective second kind integral equations have weakly singular kernels and can therefore be treated by various numerical methods. The operators are no ordinary pseudodifferential operators, however, L being parabolic, and therefore little is known about the first kind equations.

5 Convergence

The state of the art of asymptotic error estimates for BEM is described in several detailed articles by W. L. Wendland, contained and quoted in the books mentioned in section 1.4. In particular, [25] and [28] give a complete survey of the known results, ranging from abstract approximation schemes for general pseudodifferential operators to fully discretized boundary integral equations including numerical integration methods. [28] contains tables showing relations which have to be satisfied by the approximation orders of the coordinate representation of Γ and the approximation orders in the numerical integrations in order to get highest convergence rates.

Here are some remarks on the basic rules of error estimates for BEM. They are generalizations of those for FEM, and are particularly easy to state for Galerkin methods, so we consider only these here.

Let $A : H^\alpha(\Gamma) \rightarrow H^{-\alpha}(\Gamma)$ be a bijective and continuous linear operator for some real α . Assume that A satisfies a *Gårding inequality*:

There exists a compact operator $T : H^\alpha(\Gamma) \rightarrow H^{-\alpha}(\Gamma)$ and a $\gamma > 0$ such that

$$\operatorname{Re}(v, (A + T)v) \geq \gamma \|v\|_{H^\alpha(\Gamma)}^2 \quad \forall v \in H^\alpha(\Gamma). \quad (5.1)$$

If A is a pseudodifferential operator of order 2α , then (5.1) is equivalent to the strong ellipticity of A . The first kind boundary integral operators for the Dirichlet respectively Neumann problem for second order strongly elliptic systems (the operators V resp. D in potential theory) satisfy (5.1) with $\alpha = -\frac{1}{2}$ resp. $\alpha = \frac{1}{2}$, even on domains with corners and edges.

For the Galerkin procedure, one has a sequence of finite-dimensional subspaces

$$H_h \subset H^\alpha(\Gamma) \quad (0 < h < h_0), \quad (5.2)$$

and one looks for $u_h \in H_h$ satisfying

$$(v, Au_h) = (v, f) \quad \forall v \in H_h, \quad (5.3)$$

where $f = Au \in H^{-\alpha}(\Gamma)$.

If for all $u \in H^\alpha(\Gamma)$,

$$\inf\{\|u - \tilde{u}\|_{H^\alpha(\Gamma)} \mid \tilde{u} \in H_h\} \longrightarrow 0 \quad \text{as } h \rightarrow 0,$$

then (5.1) implies *stability* and *quasioptimal convergence* for the Galerkin procedure:

There exists $h_1 > 0$ such that for all $h < h_1$, (5.3) defines a unique $u_h \in H_h$, and there is a constant c independent of u and h with

$$\|u_h\|_{H^\alpha(\Gamma)} \leq c\|u\|_{H^\alpha(\Gamma)}$$

and

$$\|u - u_h\|_{H^\alpha(\Gamma)} \leq c \inf\{\|u - \tilde{u}\|_{H^\alpha(\Gamma)} \mid \tilde{u} \in H_h\}. \quad (5.4)$$

Now, as is well known e.g. from FEM for shells, the boundary element trial spaces H_h as described in 2.2.4 above, have the following *approximation property*:

There are numbers k and m such that for any $s < t$ with $s < m$ and $t \leq k$ and any $u \in H^t(\Gamma)$ there exists an approximating $\tilde{u} \in H_h$ with

$$\|u - \tilde{u}\|_{H^s(\Gamma)} \leq ch^{t-s}\|u\|_{H^t(\Gamma)} \quad (5.5)$$

where c does not depend on u or h . m measures the interelement differentiability and k the degree of the polynomials.

This together with (5.4) gives a convergence order

$$\|u - u_h\|_{H^\alpha(\Gamma)} = \mathcal{O}(h^{t-\alpha}) \quad \text{if } u \in H^t(\Gamma). \quad (5.6)$$

Using an inverse assumption and duality arguments, one can estimate the error also in norms different from the “energy norm” $H^\alpha(\Gamma)$:

$$\|u - u_h\|_{H^s(\Gamma)} = \mathcal{O}(h^{t-s}) \quad \text{if } u \in H^t(\Gamma), \quad (5.7)$$

where s is restricted by

$$2\alpha - k \leq s < m.$$

Note that for (5.6) to hold, one needs $\alpha < m$. This is just the conformity condition (5.2). In particular, if $\alpha \leq 0$, one can use discontinuous trial functions.

The highest order of convergence is obtained for $s = 2\alpha - k$ and $t = k$:

$$\|u - u_h\|_{H^{2\alpha-k}(\Gamma)} = \mathcal{O}(h^{2(k-\alpha)}) \quad \text{if } u \in H^k(\Gamma). \quad (5.8)$$

Here it is an advantage to have *negative* α . Then, however, the L^2 -condition number of the discrete equations blows up like $\mathcal{O}(h^{-2|\alpha|})$. This reflects the fact that equations with such operators are traditionally called *ill-posed*. It depends on the size of the linear system and the machine precision, which of the two effects, the theoretical Galerkin error or the round-off error magnified by a big condition number, becomes dominant. If the size of the system is chosen so that both effects balance, then the approximation is of the same order as for approximation methods specifically created for ill-posed problems, e.g. Tychonov regularization.

When the approximate solution of the boundary integral equations is inserted into the representation formula, then the highest convergence rate (5.8) on the boundary is inherited by the solution of the boundary value problem away from the boundary:

One has for the multiple layer potentials K_k from 3.2:

$$\|K_k\varphi\|_{H^s(\Omega_0)} \leq c\|\varphi\|_{H^t(\Gamma)} \quad \text{for any } s \text{ and } t$$

for bounded subdomains Ω_0 with $\overline{\Omega_0} \subset R^n \setminus \Gamma$.

A similar analysis can be given for the least squares method, because this is identical to the Galerkin method for the operator A^*A . Thus α has to be replaced by 2α , and then (5.1) is always satisfied.

For collocation methods, the corresponding analysis is available for two-dimensional problems.

6 Singularities

In BEM one has to deal with singularities in two places: In the numerical computation of integrals with singular kernels, and in the case of non-smooth boundaries or boundary conditions, with singularities of the solution that cause large approximation errors.

6.1 Singular integrals

The numerical approximation of singular integrals has been studied since a long time, but the results so far do not cover BEM completely. Thus in the preparation of every BEM application program, this point seems to be a challenge to skill and phantasy. There is a general recipe, however, namely to subtract the main singularity and compute the corresponding integrals analytically if possible. For the remaining integrals one uses quadrature formulas that are, if necessary, adapted to the still present weak singularities. The main singularities frequently are contained in a kernel of convolutional structure. Especially in two-dimensional problems the resulting translation invariance allows very useful simplifications.

If hypersingular integrals appear, they can sometimes be reduced to weakly singular ones by partial integration. For example in R^2 , the bilinear form that defines the matrix elements in the Galerkin method (2.16) for the first kind integral equation (2.10) with the hypersingular integral operator D , is equal to the same bilinear form for the weakly singular integral operator V , evaluated with the first derivatives of the respective trial functions. A similar idea works also for the equation (4.3) in R^3 [18].

A review of the state of the analysis of numerical integration in BEM is given in [28].

6.2 Non-smooth boundaries

The convergence rates in (5.8) are limited by two numbers: The degree of the polynomials chosen as a basis for the boundary element functions, and the Sobolev index describing the regularity of the solution u . The first one is a question of algorithmic complexity.

The second one, the highest value of t such that $u \in H^t(\Gamma)$, is no problem for smooth domains: From the regularity theory for elliptic pseudodifferential operators one knows $u \in H^t(\Gamma)$, as soon as $f = Au \in H^{t-2\alpha}(\Gamma)$. Frequently the right hand sides are even in $C^\infty(\Gamma)$, e.g. in scattering problems, so any t is allowed.

If corners and edges are present, then the solution has singularities there. Consider for instance the screen scattering problem of section 4.1 or a crack problem in elasticity. There it is known that the solution u behaves like $\rho^{1/2}$ and $\partial_n u$ like $\rho^{-1/2}$ near the edge, where ρ denotes the distance to the edge. Therefore u is contained at most in $H^{1-\epsilon}(\Gamma)$ for any $\epsilon > 0$, but not in $H^1(\Gamma)$. Then the Galerkin method for equation (4.3) has from (5.8) with $\alpha = 1$ a highest convergence rate of $\mathcal{O}(h^{1-2\epsilon})$.

If one knows the precise shape of the singularities, one can design special boundary element functions in order to obtain a better approximation than given by (5.5). Suppose it is known that

$$u = \beta u^s + u^0, \quad (6.1)$$

where u^s is explicitly known, only the constant β and u^0 depend on the given data, and u^0 is regular: $u^0 \in H^t(\Gamma)$. Then one can take trial functions of the form

$$\tilde{u} = \tilde{\beta} u^s + \tilde{u}^0, \quad (6.2)$$

with the same u^s and regular boundary elements \tilde{u}^0 .

Then for the best approximation, $\beta = \tilde{\beta}$, and hence

$$\|u - \tilde{u}\|_{H^s(\Gamma)} \leq ch^{t-s} \|u^0\|_{H^t(\Gamma)}.$$

Thus the convergence rate is solely determined by the regular part u^0 .

Decompositions as in (6.1) are known for two-dimensional domains with corners [13], where the singularities have the form

$$u^s = \sum_{j=1}^J \sum_{k=0}^{k_j} c_{jk} \rho^{\alpha_j} (\log \rho)^k.$$

Here ρ is the distance to the corner points and α_j are some complex numbers depending on the corner angles and the ‘‘Mellin symbol’’ of the integral operator. They are given as zeros of some analytic function, the multiplicities being $k_j + 1$. As an example, in the Dirichlet and Neumann problem in potential theory, α_j has the form

$$\alpha_j = \frac{l\pi}{\omega_i} + m,$$

where l and m are integers and ω_i is the angle at the i -th corner point.

Singularities in three-dimensional problems are more complicated and are the subject of a currently very active field of research. As an example, consider the screen scattering or crack problem as above. There one has (6.1) with

$$u^s = \rho^{1/2}$$

with ρ the distance to the edge, but now $\beta = \beta(s)$ is a function of the parameter s on the edge curve. Thus β itself has to be approximated by one-dimensional finite element functions on this curve. One can show [23] that for smooth data one has $u^0 \in H^{2-\epsilon}(\Gamma)$ for any $\epsilon > 0$. Therefore from (5.8) one can obtain a highest convergence rate of $\mathcal{O}(h^{3-2\epsilon})$ for the augmented Galerkin procedure with trial functions (6.2).

7 Coupling of FEM and BEM

If the coefficients of the differential operator are not constant, then in general a fundamental solution is not known and therefore *pure* BEM cannot be applied. Frequently, however, the inhomogeneities are contained in a subdomain Ω_1 , whereas the system in the other part Ω_2 of the domain Ω is described by a differential operator with constant coefficients whose

fundamental solution is known. For example, in problems on unbounded domains Ω the inhomogeneities are frequently appearing only in a bounded domain Ω_1 .

In this situation, a coupling of FEM on Ω_1 and BEM on Ω_2 can be used. Such couplings are popular in applications, see [9]. For the case of potential theory, also a mathematical analysis including error estimates is available. Recently, more general equations and systems, in particular those of linear elasticity, have been investigated, see [27, 26] for a survey.

Most coupling methods yield nonsymmetric matrices which is unpleasant for both the application and the error analysis. In the case of elasticity theory in [26] for example, the assumption has to be made that the ratio of the meshwidths in the domain Ω_1 for FEM and on the coupling boundary for BEM tends either to zero or to infinity in order to prove convergence.

Here I describe a symmetrical method which, in case of selfadjoint boundary value problems, yields symmetric matrices and allows a very simple error analysis. For simplicity, only the Dirichlet problem for second order strongly elliptic systems will be treated, but the general case of higher order operators and other boundary conditions as in section 3 presents no principal difficulty. More precisely, the problem is as follows:

The domain Ω is decomposed into

$$\Omega = \Omega_1 \cup \Gamma_c \cup \Omega_2,$$

where $\overline{\Gamma_c} := \partial\Omega_1 \cap \partial\Omega_2$ is the coupling boundary. We write further $\Gamma_j := \partial\Omega_j \setminus \Gamma_c$ ($j = 1, 2$), and allow Γ_1 or Γ_2 to be empty, as e.g. if Ω_1 and Ω_2 are the interior and exterior of a closed boundary Γ_c . On the other hand, also nonclosed Γ_1 , Γ_2 , and Γ_c are allowed, as e.g. if Ω_1 and Ω_2 are two adjacent rectangles.

We consider the differential equations

$$P_j u_j = f_j \quad \text{in } \Omega_j \quad (j = 1, 2) \quad (7.1)$$

with boundary conditions

$$u_j = g_j \quad \text{on } \Gamma_j \quad (j = 1, 2) \quad (7.2)$$

and the interface conditions

$$\left. \begin{aligned} u_1 &= u_2 \\ \partial_{\nu_1} u_1 &= \partial_{\nu_2} u_2 \end{aligned} \right\} \text{ on } \Gamma_c. \quad (7.3)$$

Here P_1 and P_2 are second order (matrix-valued) operators as in (3.6), and ∂_{ν_1} and ∂_{ν_2} are the respective conormal derivatives, which may be different. The normal vector is exterior to Ω_1 and interior to Ω_2 . The given data are f_j on Ω_j and g_j on Γ_j . For simplicity, we assume that $f_2 \equiv 0$, $g_1 \equiv 0$ and that P_2 contains no first order differential operators. More essentially, we also require that a fundamental solution G_2 for P_2 is known, so that in Ω_2 we have the representation formula (3.10)

$$u_2(x) = \int_{\partial\Omega_2} \left((\widetilde{\partial_{\nu(y)}} G(x, y))^{\top} v(y) - G(x, y) \varphi(y) \right) do(y), \quad (7.4)$$

where $v := u_2|_{\partial\Omega_2}$, $\varphi := \partial_{\nu_2} u_2|_{\partial\Omega_2}$. Taking Cauchy data in (7.4), one finds the relations on $\partial\Omega_2$

$$v = -A\varphi + Bv \quad (7.5)$$

$$\varphi = C\varphi - Dv \quad (7.6)$$

which define the integral operators A, B, C, D , constituting the Calderón projector on Ω_2 . In the case of potential theory, we have with the notation of section 2

$$\begin{pmatrix} -A & B \\ C & -D \end{pmatrix} = \begin{pmatrix} -V & \frac{1}{2} + K \\ \frac{1}{2} - K' & -D \end{pmatrix} .$$

On Ω_1 we have the first Green formula (3.7)

$$\Phi_{\Omega_1}(u_1, w) - \int_{\partial\Omega_1} (\partial_{\nu_1} u_1)^\top w \, do = \int_{\Omega_1} f_1^\top w \, dx \quad (7.7)$$

for all $w \in H^1(\Omega_1)$. If we assume that u_1 satisfies the boundary condition (7.2) on Γ_1 and the test function w satisfies the corresponding homogeneous boundary condition

$$w \in H_0^1(\Omega_1) := \{w \in H^1(\Omega_1) \mid w|_{\Gamma_1} = 0\}, \quad (7.8)$$

then (7.7) is just the weak formulation of the boundary value problem on Ω_1 , if $\partial_{\nu_1} u_1$ is given on Γ_c .

Now the common coupling method [9] can be described as follows: Solve (numerically) one of the equations (7.5), (7.6) for φ in terms of v and insert the resulting expression for $\partial_{\nu_2} u_2|_{\Gamma_c} = \partial_{\nu_1} u_1|_{\Gamma_c}$ in terms of $u_1|_{\Gamma_c}$ into (7.7). Then (7.7) has a familiar variational form to which FEM can be applied. This works particularly nice if a *Green function* G_2 for Ω_2 is known. Then (7.6) is simply

$$\varphi = -Dv,$$

and inserting this into (7.7) gives

$$\Phi_{\Omega_1}(u_1, w) + \int_{\Gamma_c} w^\top D(u_1|_{\Gamma_c}) \, do = \int_{\Omega_1} f_1^\top w \, dx - \int_{\Gamma_c} w^\top Dg_2 \, do. \quad (7.9)$$

Here the bilinear form on the left hand side satisfies a Gårding inequality in $H^1(\Omega_1)$ if P_1 and P_2 are strongly elliptic, and it is symmetric and positive definite if P_1 and P_2 are in addition selfadjoint. Thus one obtains immediately that every conforming Galerkin method for (7.9) converges with optimal order. This method is successfully applied e.g. in plasma physics, see [1].

If a Green function for Ω_2 is not known then one introduces

$$\varphi = \partial_{\nu_2} u_2|_{\partial\Omega_2}$$

as an additional unknown. The common method then takes a weak formulation of the *first* equation(7.5) on the boundary together with (7.7) and the coupling conditions (7.3) to form a system which is discretized by approximating u_1 with finite elements in the domain Ω_1 and φ with boundary elements on the boundary $\partial\Omega_2$. The resulting matrix is not symmetric, and one does not have a Gårding inequality except for scalar equations.

One obtains a symmetrical method if one adds to (7.7) weak forms of *both* equations (7.5) and (7.6) on the boundary. Thus we define the following bilinear form

$$\begin{aligned} a(u, \varphi; w, \psi) &:= \Phi_{\Omega_1}(u, w) \\ &+ \int_{\Gamma_c} \left\{ w^\top D(u|_{\Gamma_c}) - w^\top C\varphi - \psi^\top u + \psi^\top B(u|_{\Gamma_c}) \right\} \, do \\ &- \int_{\partial\Omega_2} \psi^\top A\varphi \, do \end{aligned} \quad (7.10)$$

for $u, w \in H_0^1(\Omega_1)$; $\varphi, \psi \in H^{-\frac{1}{2}}(\partial\Omega_2)$. The given data $f_1 \in H^{-1}(\Omega_1)$, $g_2 \in \tilde{H}^{\frac{1}{2}}(\Gamma_2)$ (note $f_2 = 0$, $g_1 = 0$) define the linear form

$$\ell(w, \psi) := \int_{\Omega_1} f_1^\top w \, dx + \int_{\Gamma_2} \psi^\top g_2 \, do - \int_{\partial\Omega_2} \psi^\top B g_2 \, do - \int_{\Gamma_c} w^\top D g_2 \, do. \quad (7.11)$$

We consider the variational equation for $(u, \varphi) \in H_0^1(\Omega_1) \times H^{-\frac{1}{2}}(\partial\Omega_2)$

$$a(u, \varphi; w, \psi) = \ell(w, \psi) \quad \forall (w, \psi) \in H_0^1(\Omega_1) \times H^{-\frac{1}{2}}(\partial\Omega_2). \quad (7.12)$$

The corresponding Galerkin scheme is:

Find $(u_h, \varphi_h) \in H_h^1 \times H_h^{-\frac{1}{2}}$ such that

$$a(u_h, \varphi_h; w, \psi) = \ell(w, \psi) \quad \forall (w, \psi) \in H_h^1 \times H_h^{-\frac{1}{2}}. \quad (7.13)$$

We can then show the following theorem, and we give a proof here, because the method is apparently new.

Theorem 7.1 *Let P_1 and P_2 be strongly elliptic, let P_2 have a selfadjoint principal part, and let the problem (7.1)–(7.3) be uniquely solvable. Then the weak solution of (7.12) exists and defines a solution of (7.1)–(7.3), where $u_1 = u$ and u_2 is given by (7.4). Furthermore every Galerkin scheme (7.13) with approximating finite dimensional spaces $H_h^1 \subset H_0^1(\Omega_1)$, $H_h^{-\frac{1}{2}} \subset H^{-\frac{1}{2}}(\partial\Omega_2)$ converges with optimal order. If P_1 and P_2 are selfadjoint, then so are the bilinear form a in (7.12) and the stiffness matrix of the Galerkin scheme (7.13).*

Proof. We assume everything to be real-valued, and we make a simplifying assumption which in general is satisfied modulo compact perturbations:

P_2 is selfadjoint, and the Dirichlet problem in Ω_2 is solvable.

This implies that the operators A and D are selfadjoint and that the following lemma holds (which is evident in potential and elasticity theory):

Lemma 7.2 *Under the present assumptions, the adjoint B' of the operator B in (7.5) is $1 - C$ with C from (7.6).*

We omit the proof which uses the first Green formula in Ω_2 .

Now assume that (7.12) holds. For $w = 0$ this is

$$\int_{\partial\Omega_2} \psi^\top \{(B - 1)v - A\varphi\} do = 0 \quad \forall \psi \in H^{-\frac{1}{2}}(\partial\Omega_2),$$

where

$$v = \begin{cases} u & \text{on } \Gamma_c \\ g_2 & \text{on } \Gamma_2. \end{cases}$$

Thus v and φ satisfy the relation (7.5) and hence are Cauchy data of the function u_2 defined by (7.4). Hence also the relation (7.6) is satisfied.

From this we find

$$\begin{aligned} a(u, \varphi; w, \psi) - \ell(w, \psi) &= \Phi_{\Omega_1}(u, w) + \int_{\Gamma_c} \{w^\top D(u_{\Gamma_c}) - w^\top (\varphi + Dv)\} do \\ &\quad - \int_{\Omega_1} f_1^\top w \, dx + \int_{\Gamma_c} w^\top D g_2 \, do \\ &= \Phi_{\Omega_1}(u, w) - \int_{\Gamma_c} w^\top \varphi \, do - \int_{\Omega_1} f_1^\top w \, dx. \end{aligned}$$

Thus (7.7) holds for all $w \in H_0^1(\Omega_1)$, with $\partial_{\nu_1} = \varphi$ on Γ_c . Hence with $u_1 = u$ we have found a solution of (7.1)–(7.3).

The selfadjointness of the bilinear form a follows from that of Φ_{Ω_1} and of the operators A and D and from the lemma.

For the convergence of the Galerkin method we show Babuška's inf-sup-condition. If (u, ϕ) is in the trial space then $(u, -\phi)$ is in the test space, and we have with the lemma

$$\begin{aligned} a(u, \varphi; u, -\varphi) &= \Phi_{\Omega_1}(u, u) + \int_{\Gamma_c} u^\top D(u|_{\Gamma_c}) do + \int_{\partial\Omega_2} \varphi^\top A\varphi do \\ &\geq c \left(\|u\|_{H^1(\Omega_1)}^2 + \|\varphi\|_{H^{-\frac{1}{2}}(\Omega_1)}^2 \right), \end{aligned}$$

because under the present assumptions the bilinear form Φ_{Ω_1} and the operators D and A are selfadjoint and positive.

References

- [1] R. Albanese, J. Blum, and O. De Barbieri. On the solution of the magnetic flux equation in an infinite domain. *Europhysics Conference Abstracts*, 10 D:41–44, 1986.
- [2] I. Babuška and K. Aziz. Survey lectures on the mathematical foundation of the finite element method. In A. K. Aziz, editor, *The Mathematical Foundation of the Finite Element Method with Applications to Partial Differential Equations*, pages 3–359, Academic Press, New York, 1972.
- [3] C. T. H. Baker. *The Numerical Treatment of Integral Equations*. Oxford University Press, 1977.
- [4] P. K. Banerjee et al., editors. *Developments in Boundary Element Methods, Vols 1,2,3*, Applied Science Publishers, London, 1979 ff.
- [5] C. A. Brebbia, editor. *Boundary Element Methods, Proc. of the 8th Int. Conf. on BEM, Como 1986*, Springer-Verlag, Berlin, 1986.
- [6] C. A. Brebbia, editor. *Boundary Element Techniques in Computer-Aided Engineering*, Martinus Nijhoff Publishers, Dordrecht, 1984.
- [7] C. A. Brebbia, editor. *Progress in Boundary Element Methods, Vol. 1*, Springer-Verlag, Berlin, 1981.
- [8] C. A. Brebbia, editor. *Progress in Boundary Element Methods, Vols 1,2,3,4*, Springer-Verlag, Berlin, 1981 ff.
- [9] C. A. Brebbia, J. C. F. Telles, and L. C. Wrobel. *Boundary Element Techniques*. Springer-Verlag, Berlin, 1984.
- [10] G. A. Chandler and I. G. Graham. Product integration-collocation methods for non-compact integral operator equations. 1986. to appear.
- [11] M. Costabel. *Boundary Integral Operators on Lipschitz domains: Elementary Results*. Preprint 898, TH Darmstadt, 1985. to appear.

- [12] M. Costabel. *Starke Elliptizität von Randintegraloperatoren erster Art*. Habilitationsschrift, Technische Hochschule Darmstadt, 1984.
- [13] M. Costabel and E. P. Stephan. Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximation. In W. Fiszdon and K. Wilmański, editors, *Mathematical Models and Methods in Mechanics 1981, Banach Center Publications Vol. 15*, pages 175–251, PWN – Polish Scientific Publishers, Warsaw, 1985.
- [14] M. Costabel and E. P. Stephan. *On the Convergence of Collocation Methods for Boundary Integral Equations on Polygons*. Preprint 982, TH Darmstadt, 1986. to appear in *Math. Comp.*
- [15] M. Costabel and W. L. Wendland. *Strong Ellipticity of Boundary Integral Operators*. Preprint 889, TH Darmstadt, 1985. *J. Reine Angew. Math.* 372:34–63, 1986.
- [16] J. Dieudonné. *Eléments d'Analyse*. Volume 8, Gauthier-Villars, Paris, 1978.
- [17] J. Mackerle and T. Andersson. Boundary element software in engineering. *Advances in Eng. Software*, 6:66–102, 1984.
- [18] J. C. Nedelec. Integral equations with non integrable kernels. *Integral Equations Oper. Theory*, 5:562–572, 1982.
- [19] B. E. Petersen. *Introduction to the Fourier Transform and Pseudodifferential Operators*. Pitman, Boston, 1983.
- [20] H. Schippers. Multigrid methods for boundary integral equations. *Numer. Math.*, 46:351–363, 1985.
- [21] E. Stephan and W. L. Wendland. Remarks to Galerkin and least squares methods with finite elements for general elliptic problems. *Manuscripta Geodaetica*, 1:93–123, 1976.
- [22] E. P. Stephan. *Boundary Integral Equations for Mixed Boundary Value Problems, Screen and Transmission Problems in R^3* . Habilitationsschrift, Technische Hochschule Darmstadt, 1984.
- [23] E. P. Stephan. Boundary integral equations for screen problems in R^3 . *Integral Equations Oper. Theory*, 9, 1986.
- [24] W. L. Wendland. Bemerkungen zu Randelementmethoden und ihren mathematischen und numerischen Aspekten. *GAMM-Mitteilungen*, 2:3–27, September 1986.
- [25] W. L. Wendland. Boundary element methods and their asymptotic convergence. In P. Filippi, editor, *Theoretical Acoustics and Numerical Techniques, CISM Courses 277*, pages 135–216, Springer-Verlag, Wien, New York, 1983.
- [26] W. L. Wendland. *On Asymptotic Error Estimates for Combined BEM and FEM*. Lecture Notes, Udine, September 1986.
- [27] W. L. Wendland. On asymptotic error estimates for the combined boundary and finite element method. In R. P. Shaw et al., editors, *Innovative Numerical Methods in Engineering*, pages 55–70, Springer-Verlag, Berlin, 1986.

- [28] W. L. Wendland. On some mathematical aspects of boundary element methods for elliptic problems. In J. Whiteman, editor, *Mathematics of Finite Elements and Applications V*, pages 193–227, Academic Press, London, 1985.
- [29] J. Whiteman, editor. *Mathematics of Finite Elements and Applications V*, Academic Press, London, 1985.