Bioelectric Forward Problems

Roadmap

- Transition from qualitative to quantitative source descriptions
  - Currents, dipoles, surface potentials, activation times
- Discuss common elements of forward problems
- Map sources to associated forward problem formulations
  - Discrete source formulations
  - Surface potentials
  - Activation time based formulation
Forward/Inverse Problems in Electrocardiography

Action Currents

- Currents that flow from action potentials and propagation
- Link the cellular sources with extracellular potentials
- Localized to regions of potential difference
- Contains a source on the front edge and a sink on the back edge
- Extracellular currents flow through volume conductor
Dipole Equivalent Sources

Dipole Sources
Dipole Source Description

Monopolar Source

\[
\phi_m = \frac{1}{4\pi\sigma} I_o \frac{1}{r}
\]

Dipolar Source

\[
\phi_d = \frac{1}{4\pi\sigma} \nabla \left( \frac{1}{r} \right) \cdot \vec{p}
\]

(For infinite medium)

\[
\phi_d = \frac{p \cos \theta}{4\pi\sigma r^2}
\]

(*For derivation, see P&B)

Equations for Distributed Sources

For dipole source not at the origin

\[
\Phi = \frac{\vec{p}}{4\pi\sigma} \cdot \nabla \left( \frac{1}{r} \right) = \frac{\vec{p}}{4\pi\sigma} \cdot \frac{1}{r^2} \hat{r}
\]

becomes

\[
\Phi = \frac{\vec{p}}{4\pi\sigma} \cdot \nabla \left( \frac{1}{|r - r'|} \right) = \frac{\vec{p}}{4\pi\sigma} \cdot \frac{1}{|r' - r|^2} \frac{r - r'}{|r' - r|^2}
\]

with

\[
\frac{1}{|r' - r|^2} = \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}
\]
**Volume Dipole Sources**

From the previous equation for a single dipole

\[
\Phi = \frac{\vec{p}}{4\pi\sigma} \cdot \nabla\left(\frac{1}{r}\right) = \frac{\vec{p}}{4\pi\sigma} \cdot \frac{1}{r^2}\hat{r}
\]

If we assume there is a volume dipole density, we can write for the extracellular field

\[
\Phi = \frac{1}{4\pi\sigma} \int_V \vec{p}_v(\vec{r}') \cdot \nabla\left(\frac{1}{r}\right) \, dV
\]

Or more generally for sources at \( p \) not on the origin

Note: the equation becomes invalid when the point \( p \) moves inside the volume.

**Surface Dipole Distributions**

![Diagram of dipole distributions on the surface with labels for Torso, Ventricle, Activation Front, and a color scale from -1800 to 2200 µV.](image)
### Surface Dipole Sources

From the previous general form of dipole volume

\[
\Phi(r) = \frac{1}{4\pi\sigma} \int_V p_v(r') \frac{r - r'}{|r - r'|^2} \, \vec{n} dV'
\]

If the dipoles are distributed on a surface, we can write for the potential

\[
\Phi(r) = \frac{1}{4\pi\sigma} \int_S p_s(r') \frac{r - r'}{|r - r'|^2} \cdot \vec{n} dS'
\]

which we can rewrite as

\[
\Phi(r) = -\frac{1}{4\pi\sigma} \int_S p_s(r') \, d\Omega_{rr'}
\]

By defining the differential solid angle as

\[
d\Omega = -\frac{r - r'}{|r - r'|^2} \cdot \vec{n} dS' = -\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \cdot \vec{n} dS'
\]

### Solid Angle

The incremental solid angle is defined as

\[
d\Omega = -\nabla \frac{1}{r} \cdot d\vec{S}
\]

which we can rewrite as

\[
d\Omega = -\frac{r - r'}{|r - r'|^2} \cdot \vec{n} dS' = -\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \cdot \vec{n} dS'
\]

and then for the total solid angle write

\[
\Omega = -\int_S \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \cdot \vec{n} dS'
\]
Uniform Surface Dipole Sources

We wrote for the potential from a surface dipole source

\[ \Phi(r) = -\frac{1}{4\pi\sigma} \int_S p_s(r') d\Omega_{r'r'} \]

If the dipole distribution is uniform, we can further simplify to

\[ \Phi(r) = -\frac{p_s \Omega}{4\pi\sigma} \]

with the usual definition for solid angle

\[ \Omega = -\int_S \frac{(r - r') \cdot \vec{n} dS'}{|r - r'|^3} \]

Myocardial Ischemia

Myocardial ischemia occurs when the blood supply to the heart is decreased, leading to cell death in the heart muscle. This can be caused by a blocked coronary artery, which can result in a heart attack.

The heart's electrical activity is disrupted, leading to abnormal heart rhythms.

[Diagram of myocardial ischemia showing normal coronary artery and areas of heart muscle with blocked arterial blood flow and resulting ischemic zones.]

[Graph showing the electrical activity of the heart in a healthy state and during ischemia.]
Solid Angle Theory and Ischemia


Adding Realistic Conditions
Primary Sources + Boundary Conditions

Returning to a previous expression for the potential from a dipole volume density in an infinite medium, we have

\[ \Phi_e(\vec{r}) = \frac{1}{4\pi\sigma_o} \int_V \vec{\rho}_o(\vec{r}) \cdot \nabla \left( \frac{1}{|\vec{r}' - \vec{r}|} \right) dV' \]

Now if we seek to determine the potential at a point on a surface to the finite, homogeneous volume conductor \( S_0 \), we can write the boundary conditions as

\[ \frac{\partial \Phi}{\partial n} \bigg|_{S_0} = 0 \]

Secondary Sources

To now incorporate the effect of the realistic boundary, picture the potential jump at the boundary \( \Phi_o \) as creating an equivalent dipole surface source \( \vec{p}_s \) \( = -\sigma_o \Phi_o \vec{n} \). This jump is necessary to ensure that potential outside the surface \( = 0 \).

We can then write for the potential at any point \( p \) inside the volume conductor

\[ \Phi_e(\vec{r}) = \frac{1}{4\pi\sigma_o} \int_V \vec{\rho}_o(\vec{r}) \cdot \nabla \left( \frac{1}{|\vec{r}' - \vec{r}|} \right) dV' - \frac{1}{4\pi\sigma_o} \int_S \sigma_o \Phi_0(\vec{r}') \vec{n} \cdot \nabla' \left( \frac{1}{|\vec{r}' - \vec{r}|} \right) \vec{n} dS \]
Potentials on the Surface

For a point on the surface, we have to integrate the expression for the secondary source and avoid the singularity at \( p \).

Then find the contribution of the avoided surface \( S_\varepsilon \) as \( \Omega = 2\pi \) and \( \Phi = \Phi / 2 \) so we can subtract \( \Phi I / 2 \) from both sides and write

\[
\Phi_e(\vec{r}) = \frac{1}{2\pi\sigma_0} \int_V \vec{p}_v(\vec{r}) \cdot \nabla \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) dV'
- \frac{1}{2\pi} \int_{S - S_\varepsilon} \Phi_0(r') \hat{n} \cdot d\Omega_{rr'}
\]

More Realistic Sources
**Epicardial Potential Source**

**Bioelectric Forward Problems**

**Green’s Theorem Formulation**

For any scalar functions $f$ and $g$:

$$\int_S (f \nabla g - g \nabla f) \cdot d\vec{A} = \int_V (f \nabla^2 g - g \nabla^2 f) dV;$$

If we select $f = \frac{1}{r}$ and $g = \phi$ and $V$ is the region between surfaces that contains no sources

$$\int_S \left(\frac{1}{r} \nabla \phi - \phi \nabla \frac{1}{r}\right) \cdot d\vec{A} = \int_V \left(\frac{1}{r} \nabla^2 \phi - \phi \nabla^2 \frac{1}{r}\right) dV,$$

$$-\int_{V_s} \nabla^2 \frac{1}{r} dV = \int_{V_s} 4\pi \delta(\vec{r} - \vec{r}^\prime) dV = \begin{cases} 0 & (p \text{ outside } V_s) \\ \frac{4\pi}{2\pi} & (p \text{ inside } V_s) \\ 0 & (p \text{ on } V_s). \end{cases}$$
Potential in the Volume

We can now take the $2\pi$ case and write the previous eqn as

$$2\pi \phi(p) = \int_{S_H} \frac{1}{r} \nabla \phi - \phi \nabla \frac{1}{r} \cdot d\hat{A}$$

Where the surface integral avoids the singularity.

By splitting the surface into heart and body surfaces, we can then rewrite this as

$$2\pi \phi(p) = \int_{S_H} \phi \, d\Omega - \int_{S_{B_H}} \phi \, d\Omega - \int_{S_H} \frac{\nabla \phi}{r} \cdot d\hat{A} + \int_{S_{B_H}} \frac{\nabla \phi}{r} \cdot d\hat{A}$$

BEM Formulation

Written once for a point on each of the two surfaces, we get

$$\phi^i_H = -\frac{1}{2\pi} \int_{S_H} \phi_H \, d\Omega^i_H + \frac{1}{2\pi} \int_{S_B} \phi_B \, d\Omega^i_{HB} + \frac{1}{2\pi} \int_{S_{B_H}} \frac{\nabla \phi_H}{r} \cdot d\hat{A} = 0$$

$$\phi^i_B = -\frac{1}{2\pi} \int_{S_H} \phi_H \, d\Omega^i_{BH} + \frac{1}{2\pi} \int_{S_B} \phi_B \, d\Omega^i_{BB} + \frac{1}{2\pi} \int_{S_{B_H}} \frac{\nabla \phi_H}{r} \cdot d\hat{A} = 0$$
Numerical Solution

Converting the previous equations to matrix form, we get

\[ P_{BB}\Phi_B + P_{BH}\Phi_H + G_{BH}\Gamma_H = 0 \]
\[ P_{HB}\Phi_B + P_{HH}\Phi_H + G_{HH}\Gamma_H = 0 \]

Which we can solve to get

\[ (P_{BB} - G_{BH}G_{HH}^{-1}P_{HB})\Phi_B = \]
\[ (G_{BH}G_{HH}^{-1}P_{HH} - P_{BH})\Phi_H \]

Transfer Coefficient Matrix

If we define a matrix of transfer coefficients,

\[ Z_{BH} = (P_{BB} - G_{BH}G_{HH}^{-1}P_{HB})^{-1} (G_{BH}G_{HH}^{-1}P_{HH} - P_{BH}) \]

We can rewrite the previous equation as

\[ \Phi_B = Z_{BH}\Phi_H \]

And then formulate an associated inverse problem

\[ \Phi_H = Z_{BH}^{-1}\Phi_B = Z_{HB}\Phi_B \]
Meaning of Transfer Coefficients

\[
\begin{pmatrix}
N_B 	imes N_H \\
N_H \times 1
\end{pmatrix}
= \begin{pmatrix}
N_B 	imes 1
\end{pmatrix}
\]

B = body  
H = heart

Sensitivity vector

Activation Wavefront Source

Torso  
Ventricle  
Activation Front
Activation Time Forward/Inverse Problems

Represent source as activation time driven uniform double layer.

Solid Angles and the Uniform Double Layer

\[ \Omega_c = 0 \]

\[ \Omega_c = \Omega_1 + \Omega_2 = 0 \]

\[ \Omega_1 = -\Omega_2 \]
### UDL for Different Shapes

<table>
<thead>
<tr>
<th>Equivalent Sources</th>
<th>Closed double layer</th>
<th>Open double layer</th>
<th>Various double layers with the same opening</th>
<th>Open double layer with two openings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double layer source</td>
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<td><img src="image2" alt="Image" /></td>
<td><img src="image3" alt="Image" /></td>
<td><img src="image4" alt="Image" /></td>
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<tr>
<td>Equivalent double layer source</td>
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<td><img src="image6" alt="Image" /></td>
<td><img src="image7" alt="Image" /></td>
<td><img src="image8" alt="Image" /></td>
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<tr>
<td>Equivalent dipole</td>
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<td><img src="image10" alt="Image" /></td>
<td><img src="image11" alt="Image" /></td>
<td><img src="image12" alt="Image" /></td>
</tr>
</tbody>
</table>

### UDL Assumptions

\[
\text{Endo} + \text{Epi} = \frac{1}{R} = 0
\]
Forward Formulation

For any time $t$, $S(t)$ of the heart is excited and we can write

$$\Phi_B(y, t) = \int_{S(t)} T(y, x) \, dx$$

or more generally

$$\Phi_B(y, t) = \int_{S(t)} T(y, x) H[t - \tau(x)] \, dx$$

which provides a way to compute potential as a function of activation time.

Summary of Sources

<table>
<thead>
<tr>
<th></th>
<th>Dipole</th>
<th>Epicardial Potentials</th>
<th>Epi-Endocardial Activation Time</th>
<th>Transmembrane Potentials</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple, few leads, conventional</td>
<td>Measurable, comprehensive, unique</td>
<td>Measurable, clinically directly useful</td>
<td>Measurable in cells or on surfaces (optical)</td>
<td></td>
</tr>
<tr>
<td>Unique with substantial constraints, not measurable, requires assumptions, misses details</td>
<td>Interpretation ambiguous, complex, ill-posed</td>
<td>Uniqueness unclear, tenuous assumptions, ill-posed</td>
<td>Not unique but well constrained, less ill posed?</td>
<td></td>
</tr>
</tbody>
</table>
Role of Anisotropy

Inhomogeneous/Isotropic

Anisotropic

Numerical Solutions of Forward Problem Models

Surface
- Geometry piecewise homogeneous and isotropic
- Integral form of equations
- Model described in terms of surfaces (numerically as triangles, quads)
- Fewer elements in the model and hence fewer equations
- Solution matrices are smaller but full

Volume
- Geometry elementwise homogenous, anisotropic
- Differential form of equations
- Model in terms of volumes (numerically as hexahedra, tetrahedra)
- More elements and hence more equations, however each equation is simpler
- Solution matrices are larger but sparse