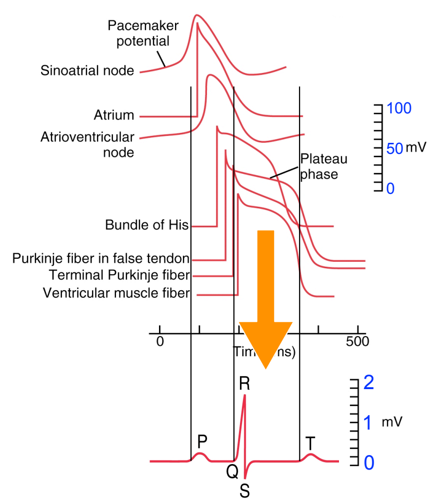
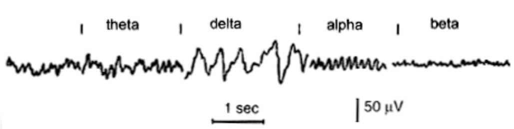
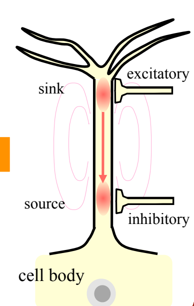
Bioelectric Forward Problems

Roadmap

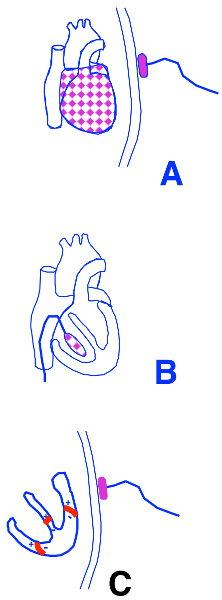


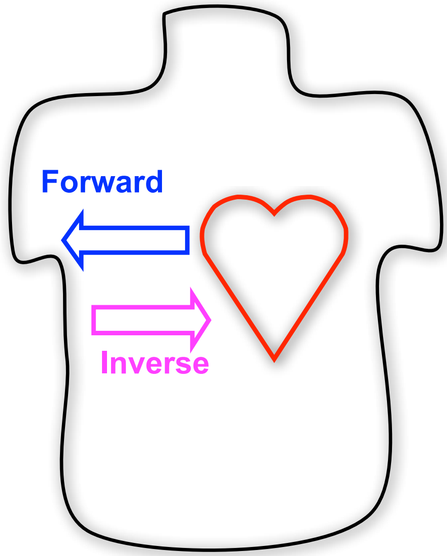
- Transition from qualitative to quantitative source descriptions
  - Currents, dipoles, surface potentials, activation times
- Discuss common elements of forward problems
- Map sources to associated forward problem formulations
  - Discrete source formulations
  - Surface potentials
  - Activation time based formulation





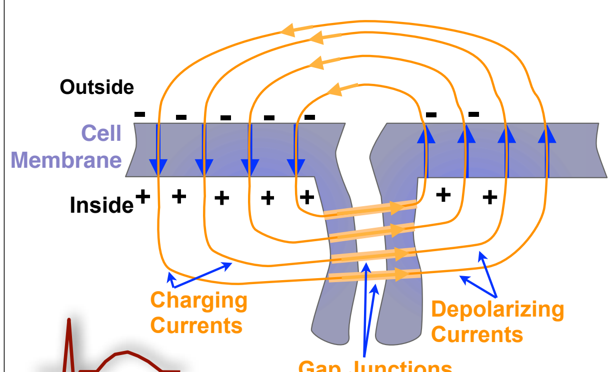
Forward/Inverse Problems in Electrocardiography

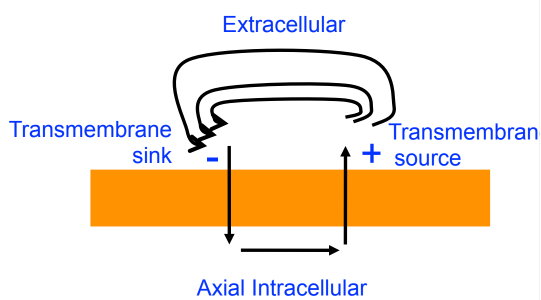




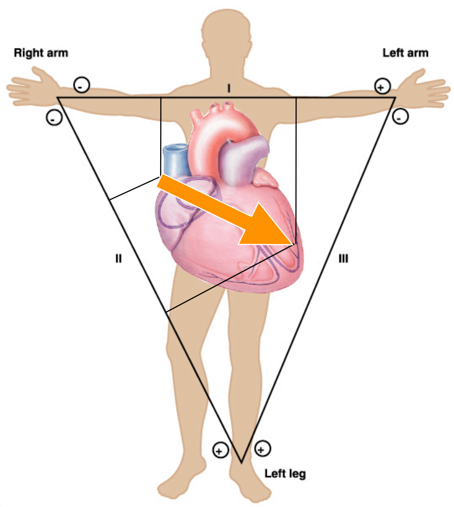
Action Currents

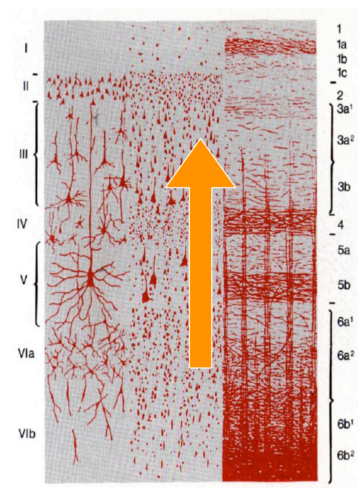
- Currents that flow from action potentials and propagation
- Link the cellular sources with extracellular potentials
- Localized to regions of potential difference
- Contains a source on the front edge and a sink on the back edge
- Extracellular currents flow through volume conductor



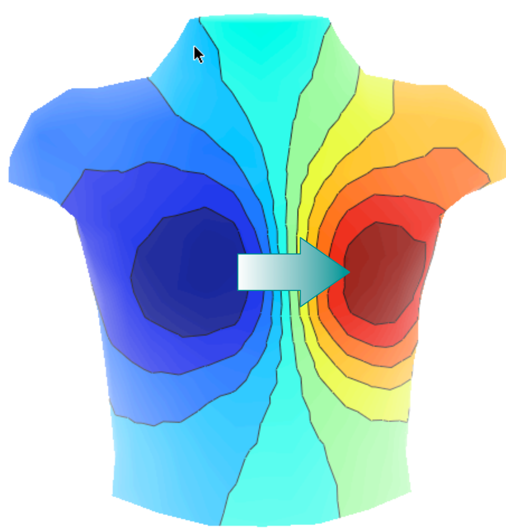


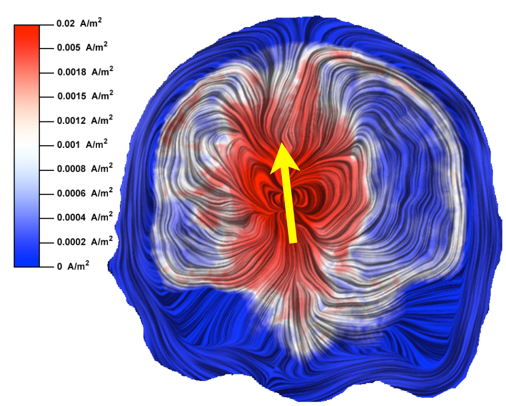
Dipole Equivalent Sources





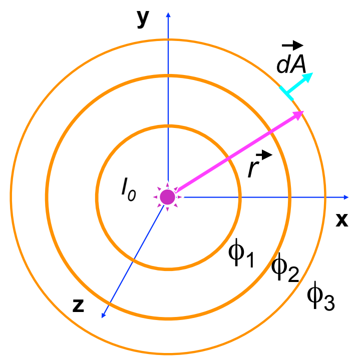
Dipole Sources





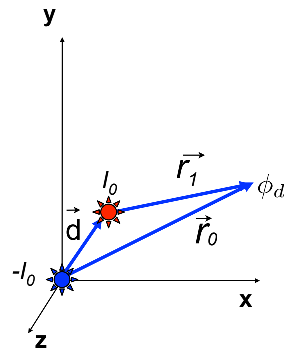
Dipole Source Description

Monopolar Source



$$\phi_m = \frac{1}{4\pi\sigma_e}\frac{I_o}{r}$$

Dipolar Source



$$\phi_d = \frac{1}{4\pi\sigma}\nabla\left(\frac{1}{r}\right)\cdot\vec{p}$$ *

$$\phi_d = \frac{p\cos\theta}{4\pi\sigma r^2}$$ (for infinite medium)

(*For derivation, see P&B)

Equations for Distributed Sources

For dipole source not at the origin

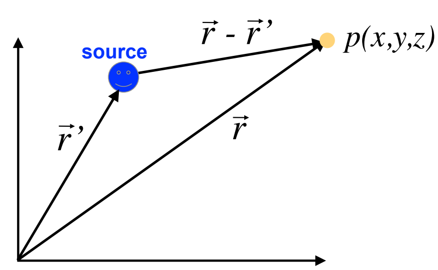
$$\Phi = \frac{\vec{p}}{4\pi\sigma}\cdot\nabla\left(\frac{1}{r}\right) = \frac{\vec{p}}{4\pi\sigma}\cdot\frac{1}{r^2}\hat{r}$$

becomes

$$\Phi = \frac{\vec{p}}{4\pi\sigma}\cdot\nabla\left(\frac{1}{|r-r'|}\right) = \frac{\vec{p}}{4\pi\sigma}\cdot\frac{1}{|\vec{r}-\vec{r}'|^2}\widehat{r-r'}$$

with

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$



Volume Dipole Sources

From the previous equation for a single dipole

$$\Phi = \frac{\vec{p}}{4\pi\sigma} \cdot \nabla(\frac{1}{r}) = \frac{\vec{p}}{4\pi\sigma} \cdot \frac{1}{r^2}\hat{r}$$

If we assume there is a volume dipole density,
we can write for the extracellular field

$$\Phi = \frac{1}{4\pi\sigma} \int_V \vec{p}_v(\vec{r}) \cdot \nabla(\frac{1}{r})\, dV$$

Or more generally for sources at p not on the origin

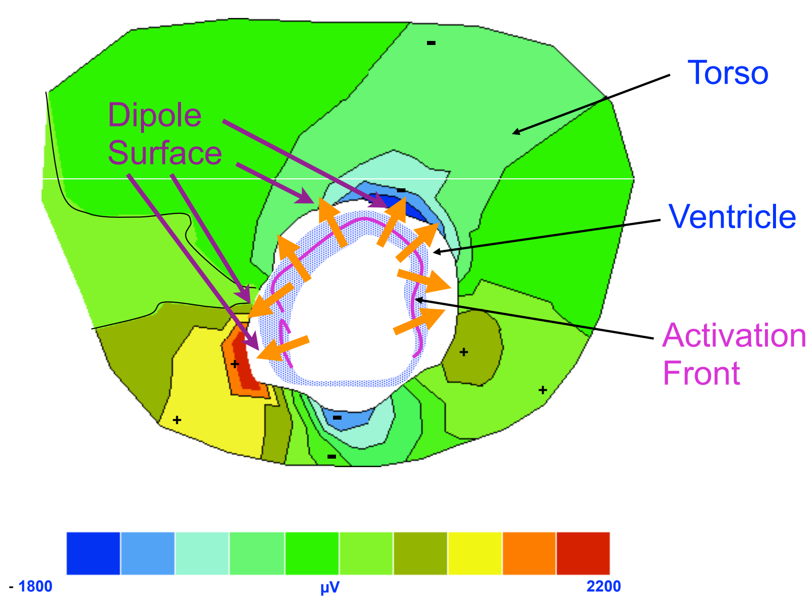
$$\Phi = \frac{1}{4\pi\sigma} \int_V \vec{p}_v(\vec{r}) \cdot \nabla \left(\frac{1}{|\vec{r} - \vec{r'}|} \right) dV'$$

or

$$\Phi(r) = \frac{1}{4\pi\sigma} \int_V p_v(\vec{r'}) \frac{\widehat{r - r'}}{|\vec{r} - \vec{r'}|^2} \vec{n} dV'$$

Note: the equation becomes invalid when the point p moves inside the volume.

Surface Dipole Distributions



Surface Dipole Sources

From the previous general form of dipole volume

$$\Phi(r) = \frac{1}{4\pi\sigma}\int_V p_v(\vec{r'})\frac{\widehat{r-r'}}{|\vec{r}-\vec{r'}|^2}\vec{n}dV'$$

If the dipoles are distributed on a surface, we can write for the potential

$$\Phi(r) = \frac{1}{4\pi\sigma}\int_S p_s(\vec{r'})\frac{\widehat{r-r'}}{|\vec{r}-\vec{r'}|^2}\cdot\vec{n}dS'$$

which we can rewrite as

$$\Phi(r) = -\frac{1}{4\pi\sigma}\int_s p_s(\vec{r'})\, d\Omega_{rr'}$$

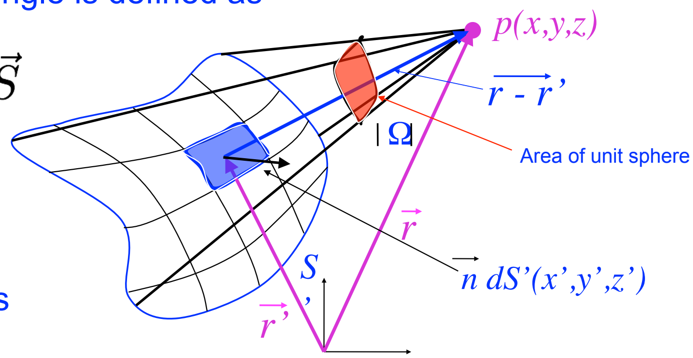
By defining the differential solid angle as

$$d\Omega = -\frac{\widehat{r-r'}}{|\vec{r}-\vec{r'}|^2}\cdot\vec{n}dS' = -\frac{(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3}\cdot\vec{n}dS'$$

Solid Angle

The incremental solid angle is defined as

$$d\Omega = -\nabla\frac{1}{r}\cdot d\vec{S}$$



which we can rewrite as

$$d\Omega = -\frac{\widehat{r-r'}}{|\vec{r}-\vec{r'}|^2}\cdot\vec{n}dS' = -\frac{(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3}\cdot\vec{n}dS'$$

and then for the total solid angle write

$$\Omega = -\int_S \frac{(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3}\cdot\vec{n}dS'$$

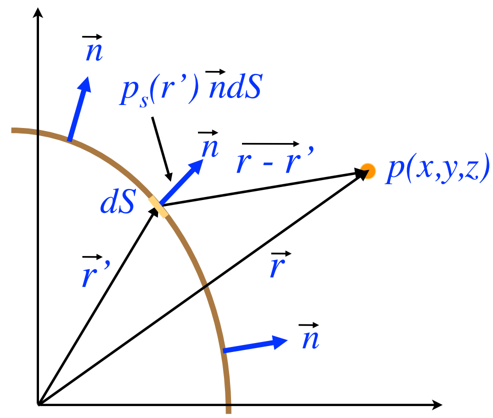
Uniform Surface Dipole Sources

We wrote for the potential from a surface dipole source

$$\Phi(r) = -\frac{1}{4\pi\sigma}\int_s p_s(\vec{r'})\, d\Omega_{rr'}$$

If the dipole distribution is uniform, we can further simplify to

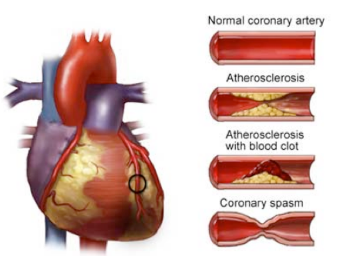
$$\Phi(r) = -\frac{p_s\Omega}{4\pi\sigma}$$

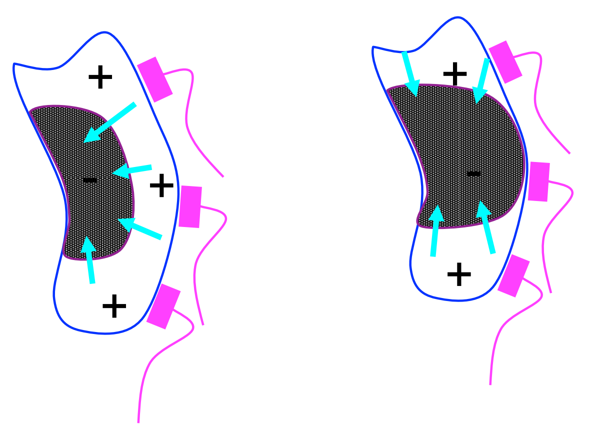


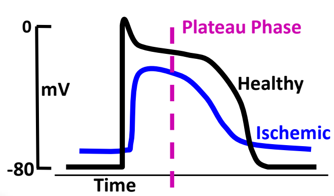
with the usual definition for solid angle

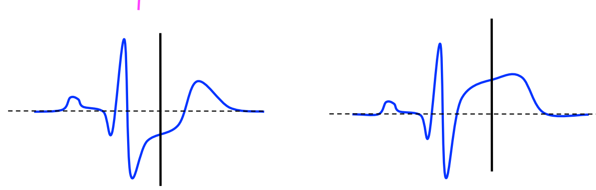
$$\Omega = -\int_S \frac{(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3}\cdot \vec{n}dS'$$

Myocardial Ischemia

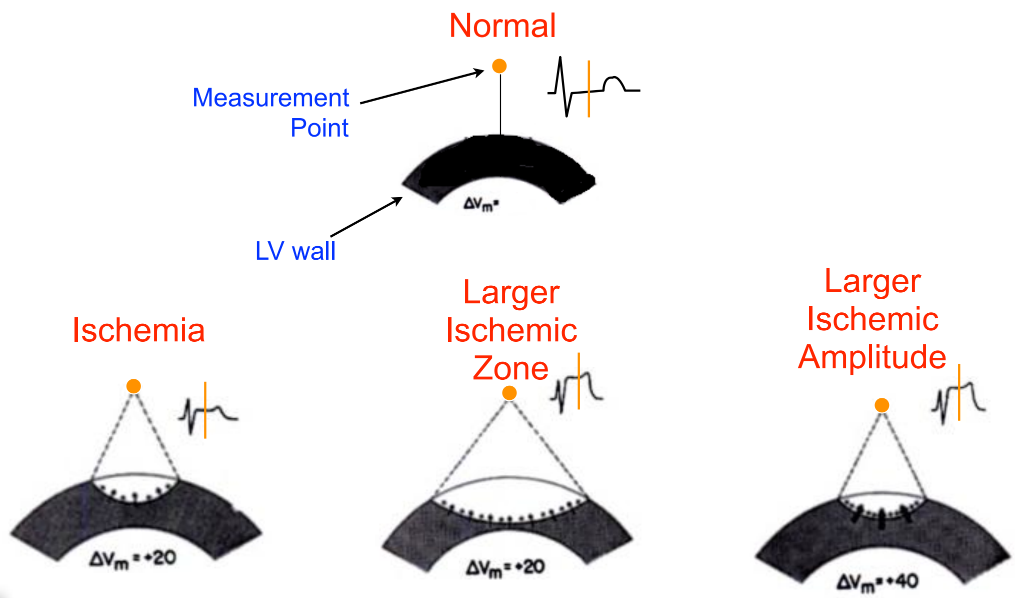






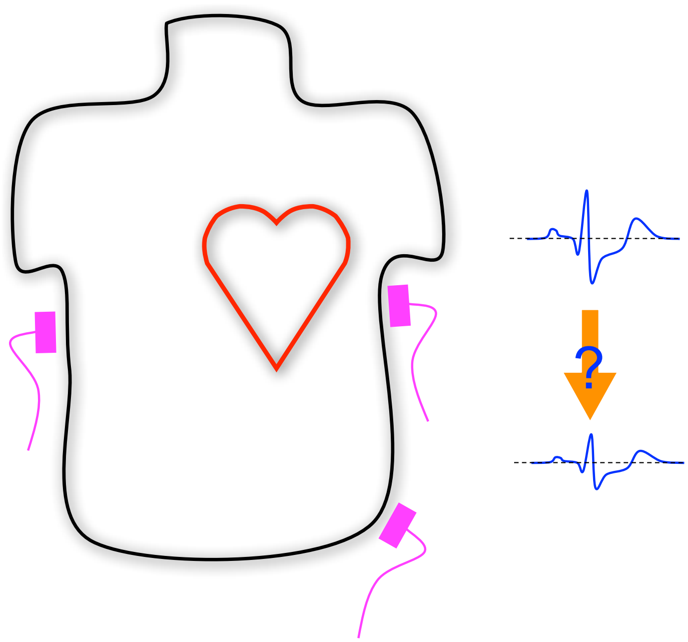


Solid Angle Theory and Ischemia



R.P. Holland and M.F. Arnsdorf, "Solid angle theory and the electrocardiogram: physiologic and quantitative interpretations", Prog. Cardiovasc. Dis. 1977 19:431-457.

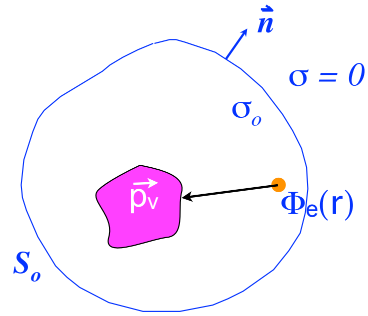
Adding Realistic Conditions



Primary Sources + Boundary Conditions

Returning to a previous expression for the potential from a dipole volume density in an infinite medium, we have

$$\Phi_e(\vec{r}) = \frac{1}{4\pi\sigma_o}\int_V \vec{p}_v(\vec{r}) \cdot \nabla\left(\frac{1}{|\vec{r}-\vec{r'}|}\right) dV'$$

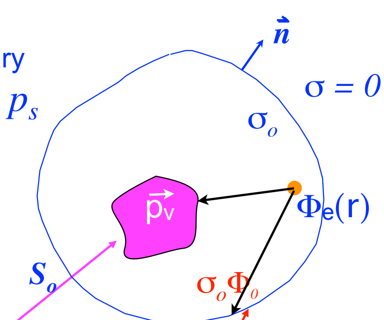


Now if we seek to determine the potential at a point on a surface to the finite, homogeneous volume conductor S_0, we can write the boundary conditions as

$$\frac{\partial \Phi}{\partial n}|_{S_o} = 0$$

Secondary Sources

To now incorporate the effect of the realistic boundary, picture the potential jump at the boundary Φ_o as creating an equivalent dipole surface source p_s = $-\sigma_o\Phi_o\boldsymbol{n}$. This jump is necessary to ensure that potential outside the surface = 0.



We can then write for the potential at any point p inside the volume conductor

Primary source

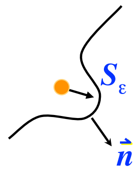
$$\Phi_e(\vec{r}) = \frac{1}{4\pi\sigma_o}\int_V \vec{p}_v(\vec{r}) \cdot \nabla\left(\frac{1}{|\vec{r}-\vec{r'}|}\right) dV' - \frac{1}{4\pi\sigma_o}\int_S \sigma_o\Phi_0(r')\hat{n} \cdot \nabla'\left(\frac{1}{|\vec{r}-\vec{r'}|}\right) \hat{n}dS$$

*Primary source assumes infinite homogenous volume conductor

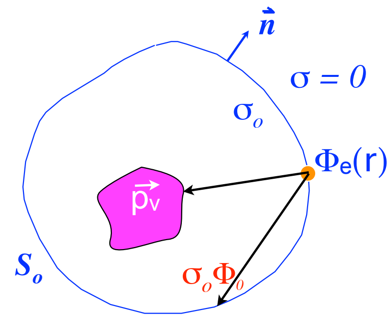
Secondary source

Potentials on the Surface

For a point on the surface, we have to integrate the expression for the secondary source and avoid the singularity at p.

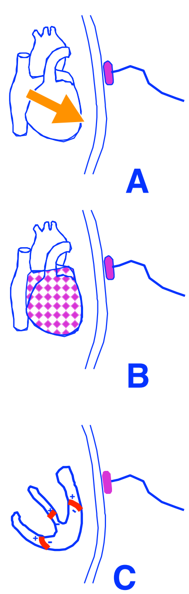


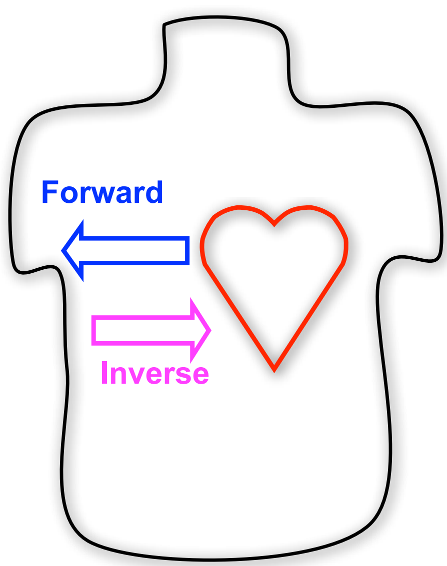
Then find the contribution of the avoided surface S_ε as $\Omega=2\pi$ and $\Phi=\Phi_o/2$ so we can subtract $\Phi_o/2$ from both sides and write



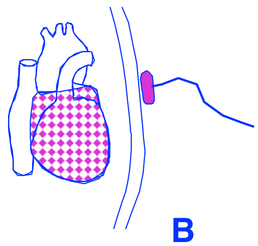
$$\Phi_e(\vec{r}) = \frac{1}{2\pi\sigma_0}\int_V \vec{p}_v(\vec{r})\cdot\nabla\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)dV' - \frac{1}{2\pi}\int_{S-S_\epsilon}\Phi_0(r')\hat{n}\cdot d\Omega_{rr'}$$

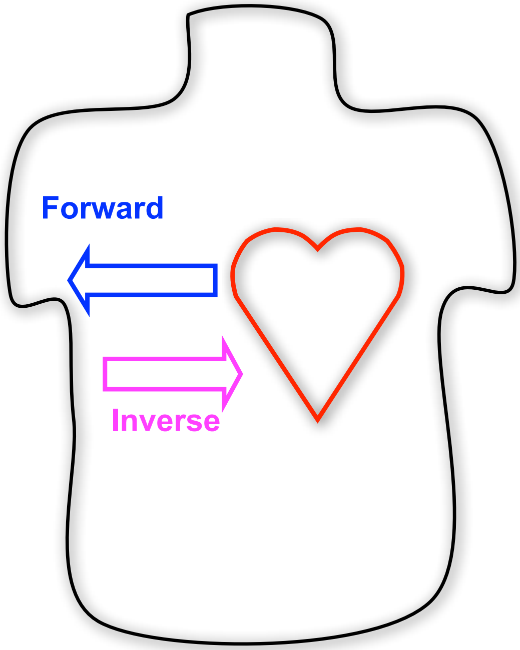
More Realistic Sources





Epicardial Potential Source





Green’s Theorem Formulation

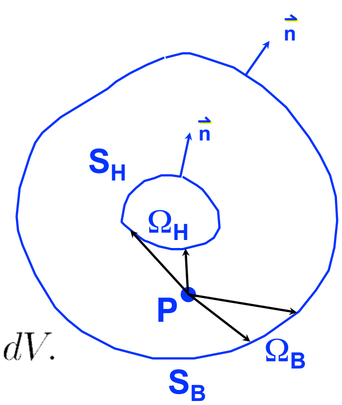
For any scalar functions *f* and *g*:

$$\int_S (f\nabla g - g\nabla f)\cdot d\vec{A} = \int_V (f\nabla^2 g - g\nabla^2 f)dV,$$

If we select f = 1/r and g = ϕ and V is the region between surfaces that contains no sources

$$\int_S (\frac{1}{r}\nabla\phi - \phi\nabla\frac{1}{r})\cdot d\vec{A} = \int_V (\underbrace{\frac{1}{r}\nabla^2\phi}_{0} - \phi\nabla^2\frac{1}{r})\, dV.$$

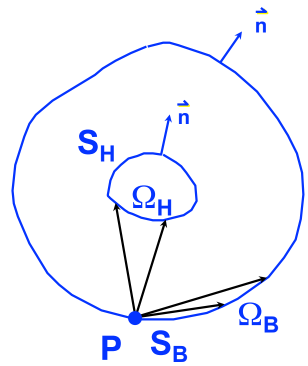
$$-\int_{V_s} \nabla^2 \frac{1}{r}\, dV = \int_{V_s} 4\pi\delta(\vec{r}-\vec{r'})\, dV = \begin{cases} 0 & (p \text{ outside } \mathrm{V_s}) \\ 4\pi & (p \text{ inside } \mathrm{V_s}). \\ 2\pi & (p \text{ on } \mathrm{V_s}). \end{cases}$$



Potential in the Volume

We can now take the 2π case and write the previous eqn as

$$2\pi\phi(p) = \int_{S_\epsilon} \left(\frac{1}{r}\nabla\phi - \phi\nabla\frac{1}{r}\right) \cdot d\vec{A}$$

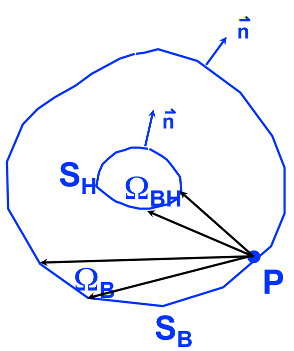


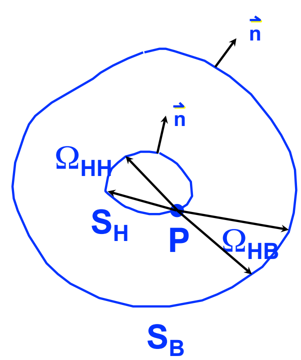
Where the surface integral avoids the singularity. By splitting the surface into heart and body surfaces, we can then rewrite this as

$$2\pi\phi(p) = \int_{S_H} \phi \, d\Omega - \int_{S_{B_\epsilon}} \phi \, d\Omega - \int_{S_H} \frac{\nabla\phi}{r} \cdot d\vec{A} + \int_{S_{B_\epsilon}} \frac{\nabla\phi}{r} \cdot d\vec{A}$$

BEM Formulation

Written once for a point on each of the two surfaces, we get





$$\phi_H^i - \frac{1}{2\pi}\int_{S_{H_\epsilon}} \phi_H \, d\Omega_{HH}^i + \frac{1}{2\pi}\int_{S_B} \phi_B \, d\Omega_{HB}^i + \frac{1}{2\pi}\int_{S_{H_\epsilon}} \frac{\nabla\phi_H}{r^i} \cdot d\vec{A} = 0$$

$$\phi_B^i - \frac{1}{2\pi}\int_{S_H} \phi_H \, d\Omega_{BH}^i + \frac{1}{2\pi}\int_{S_{B_\epsilon}} \phi_B \, d\Omega_{BB}^i + \frac{1}{2\pi}\int_{S_H} \frac{\nabla\phi_H}{r^i} \cdot d\vec{A} = 0$$

Numerical Solution

Converting the previous equations to matrix form, we get

$$P_{BB}\Phi_B + P_{BH}\Phi_H + G_{BH}\Gamma_H = 0$$

$$P_{HB}\Phi_B + P_{HH}\Phi_H + G_{HH}\Gamma_H = 0$$

Which we can solve to get

$$(P_{BB} - G_{BH}G_{HH}^{-1}P_{HB})\Phi_B =$$

$$(G_{BH}G_{HH}^{-1}P_{HH} - P_{BH})\Phi_H$$

Transfer Coefficient Matrix

If we define a matrix of transfer coefficients,

$$Z_{BH} = (P_{BB} - G_{BH}G_{HH}^{-1}P_{HB})^{-1}(G_{BH}G_{HH}^{-1}P_{HH} - P_{BH})$$

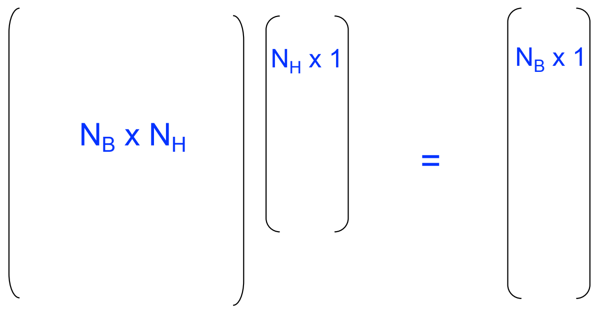
We can rewrite the previous equation as

$$\Phi_B = Z_{BH}\Phi_H$$

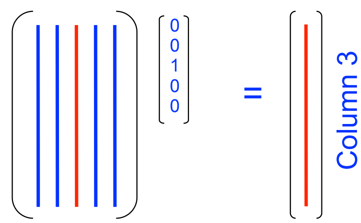
And then formulate an associated inverse problem

$$\Phi_H = Z_{BH}^{-1}\Phi_B = Z_{HB}\Phi_B$$

Meaning of Transfer Coefficients

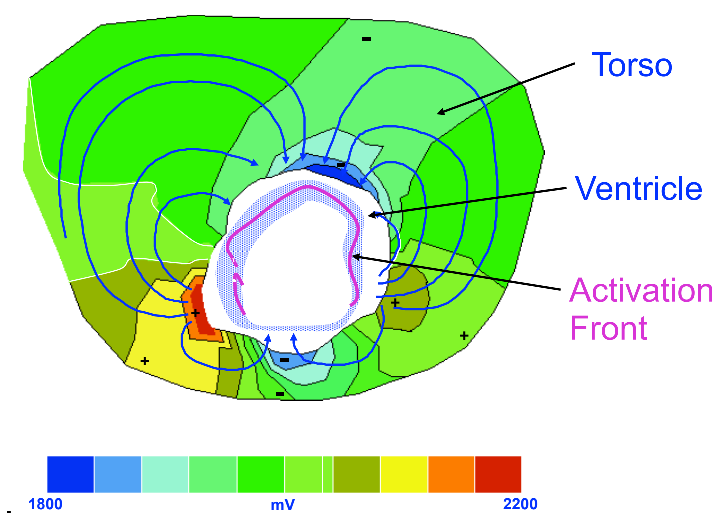


B = body
H = heart

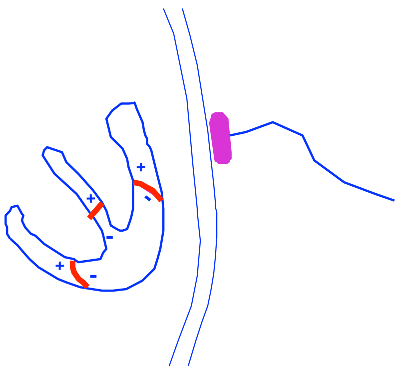


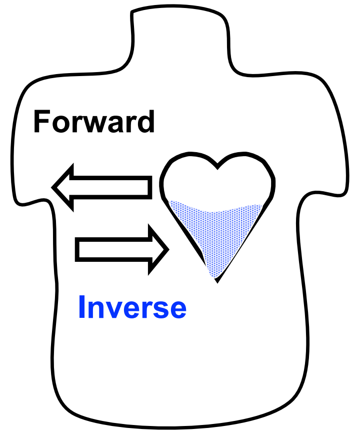
Sensitivity vector

Activation Wavefront Source



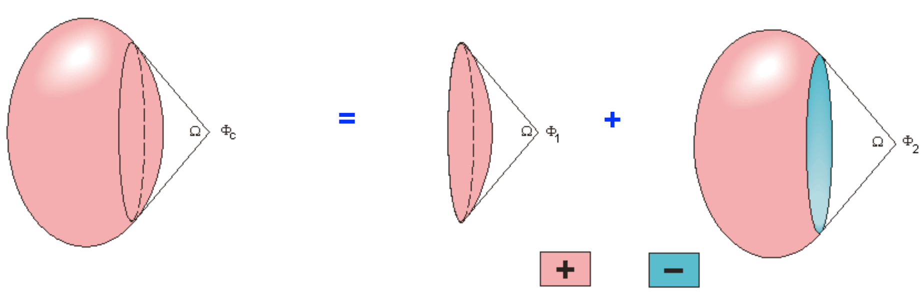
Activation Time Forward/Inverse Problems





Represent source as activation time driven uniform double layer.

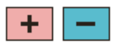
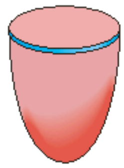
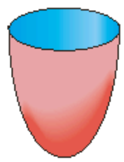
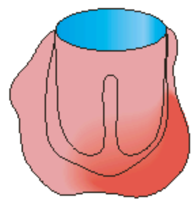
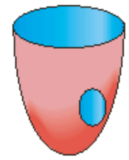
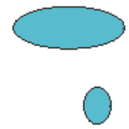
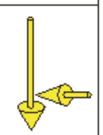
Solid Angles and the Uniform Double Layer



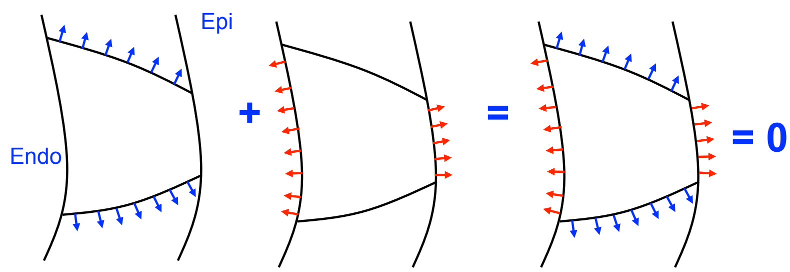
$$\Omega_c = 0$$

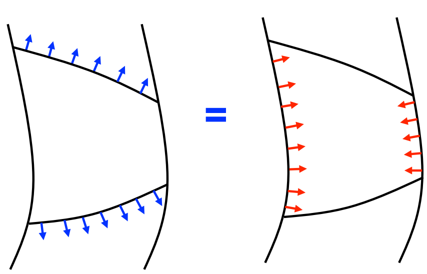
$$\Omega_c = \Omega_1 + \Omega_2 = 0$$
$$\Omega_1 = -\Omega_2$$

UDL for Different Shapes

EQUIVALENT SOURCES	TYPE OF DOUBLE LAYER SOURCES: Closed double layer	Open double layer	Various double layers with the same opening	Open double layer with two openings
Double layer source 				
Equivalent double layer source	(Zero field)			
Equvalent dipole	(Null)			

UDL Assumptions



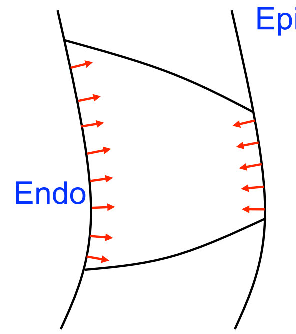


Forward Formulation

For any time t, $S(t)$ of the heart is excited and we can write

$$\Phi_B(y,t) = \int_{S(t)} T(y,x)\,dx$$

Transfer matrix

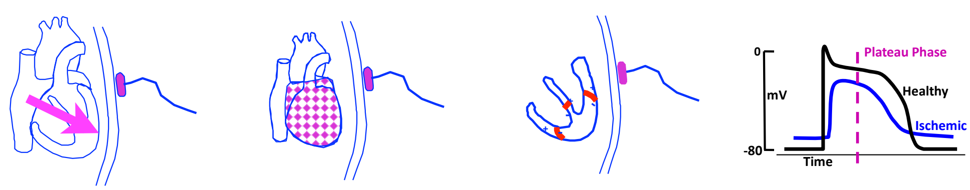


or more generally

$$\Phi_B(y,t) = \int_{S(H)} T(y,x) H[t-\tau(x)]\,dx$$

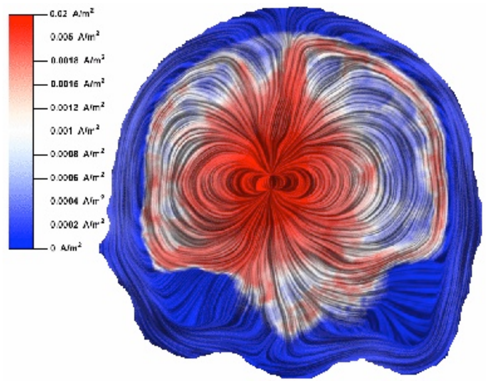
which provides a way to compute potential as a function of activation time.

Summary of Sources

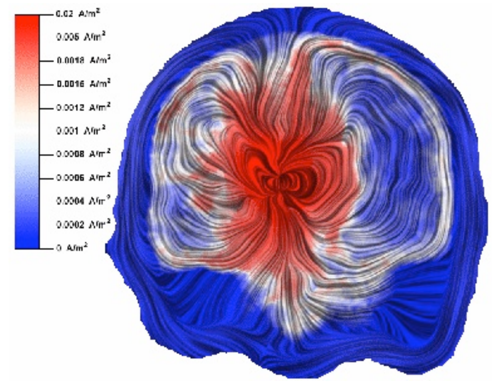


Dipole	Epicardial Potentials	Epi-Endocardial Activation Time	Transmembrane Potentials
Simple, few leads, conventional	Measurable, comprehensive, unique	Measurable, clinically directly useful	Measurable in cells or on surfaces (optical)
Unique with substantial constraints, not measurable, requires assumptions, misses details	Interpretation ambiguous, complex, ill-posed	Uniqueness unclear, tenuous assumptions, ill-posed	Not unique but well constrained, less ill posed?

Role of Anisotropy



Inhomogeneous/Isotropic



Anisotropic

Numerical Solutions of Forward Problem Models

Surface

- Geometry piecewise homogeneous and isotropic
- Integral form of equations
- Model described in terms of surfaces (numerically as triangles, quads)
- Fewer elements in the model and hence fewer equations
- Solution matrices are smaller but full

Volume

- Geometry elementwise homogenous, anisotopic
- Differential form of equations
- Model in terms of volumes (numerically as hexahedra, tetrahedra)
- More elements and hence more equations, however each equation is simpler
- Solution matrices are larger but sparse