Bioeng 6460

Electrophysiology and Bioelectricity

Modeling of Electrical Conduction in Cardiac Tissue II

Frank B. Sachse

fs@cvrti.utah.edu

CVRTI

Overview

- Partial Differential Equations
- Finite Differences Method
    - Discretization of Domains
    - Discretization of Operators
    - Discretization of Equations
- Summary

Group work

Group work

Group work

Generalized Poisson Equation for Electrical Current

$$\nabla \cdot \left(\sigma \, \nabla \Phi \right) + f = 0$$

Φ: Electrical potential [V]

σ: Conductivity tensor [S/m]

f: Current source density [A/m^3]

Scalar/ complex quantities

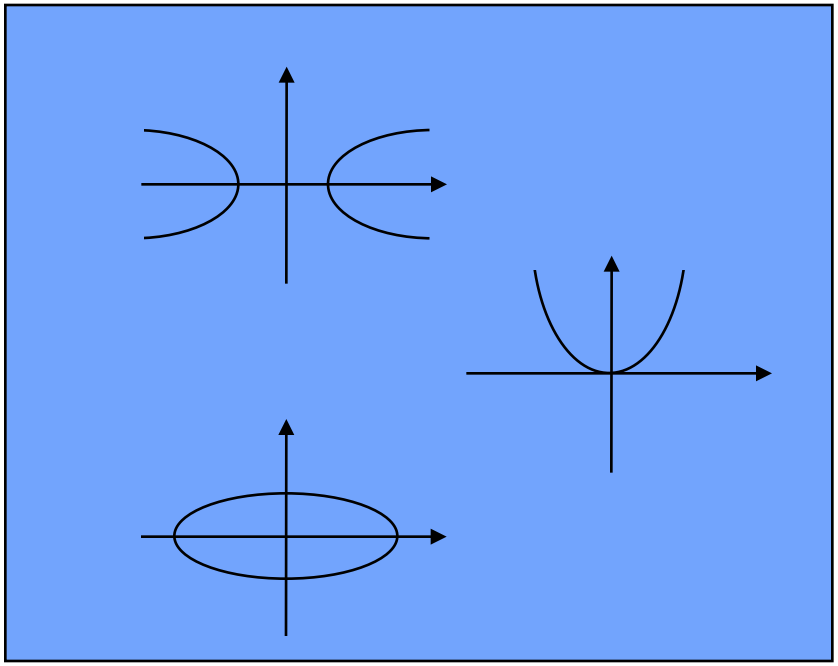
CVRTI

Classification of Partial Differential Equations

$u(x, y)$ fulfills the linear partial differential equation:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = H$$

in domain $G \subset \Re^2$



$AC - B^2 < 0$: hyperbolic

$AC - B^2 = 0$: parabolic

$AC - B^2 > 0$: elliptic

CVRTI

Group Work

Is Poisson's equation hyperbolic, parabolic, elliptic or none of those?

Assume constant scalar conductivity and a two-dimensional domain, which leads to the following simplification:

$$\sigma\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+f=0$$

CVRTI

Elliptic Partial Differential Equations

2D Poisson equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x,y)$$

2D Laplace equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

2D Helmholtz equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$$

$\rho(x,y)$: Source term

k: Constant

→ **Boundary problem**
static/(quasi-)stationary solution

CVRTI

Hyperbolic and Parabolic Differential Equations

1D wave equation - hyperbolic:

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

v: Velocity of wave propagation

1D diffusion equation - parabolic:

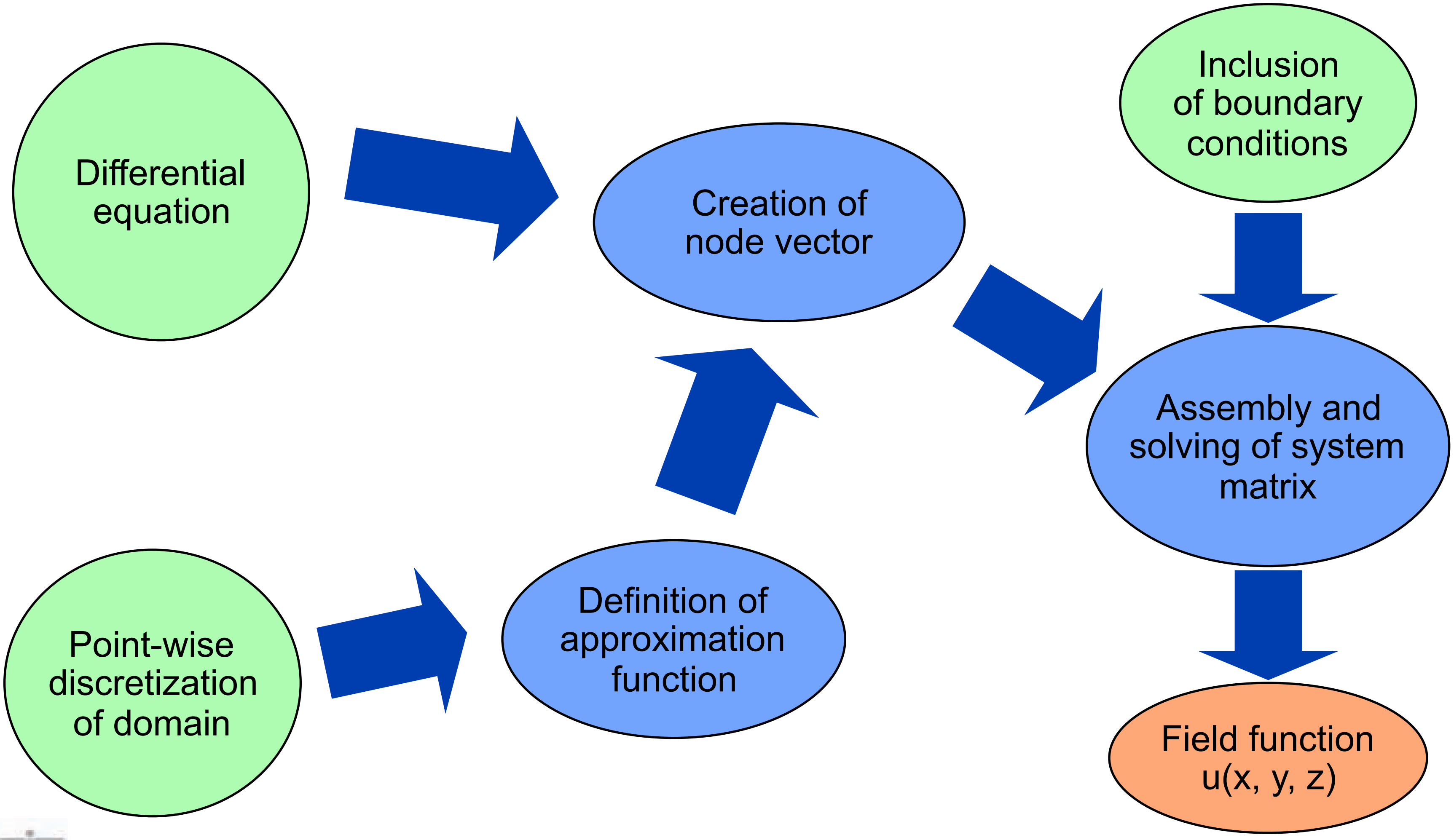
$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D \frac{\partial u}{\partial x}\right)$$

D: Diffusion coefficient

→ **Initial value problem**

CVRTI

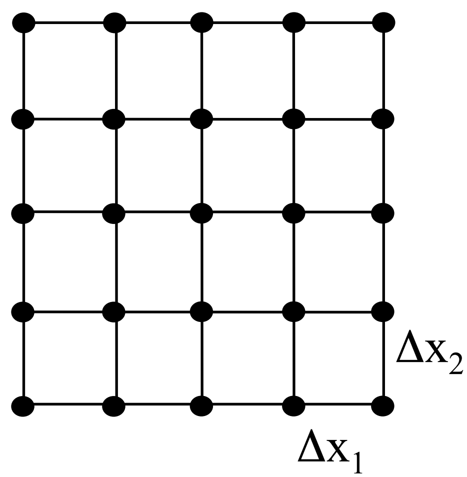
Finite Differences Method: Overview



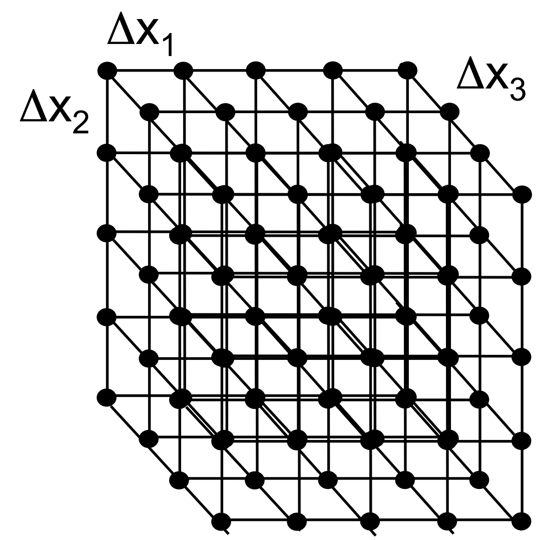
CVRTI

Spatial Discretizations: Regular Lattice

2 D



3 D

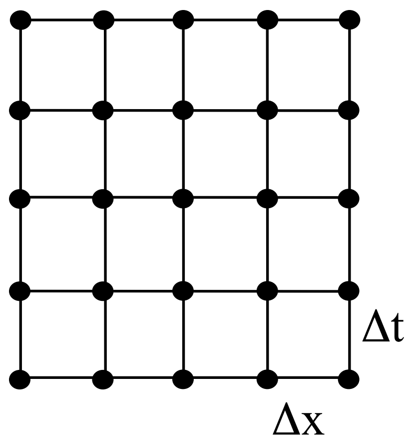


• Node, e.g. with node variables V_m, Φ_i and Φ_e

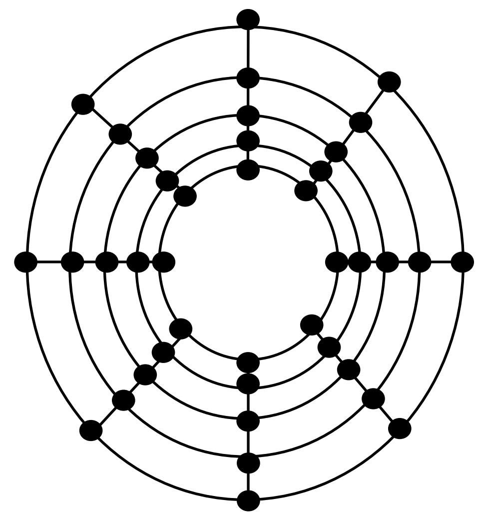
CVRTI

Exemplary Discretizations

1 D+t

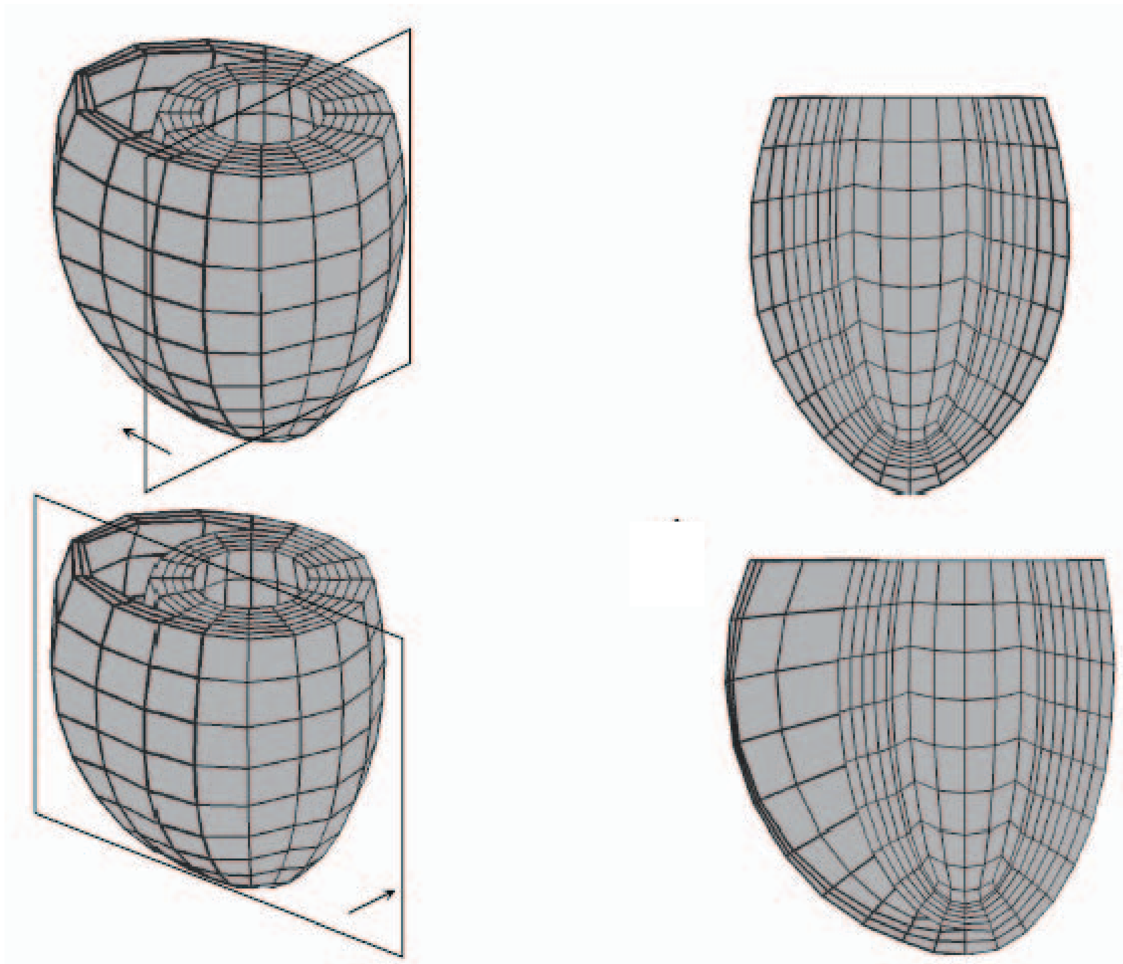


2 D



CVRTI

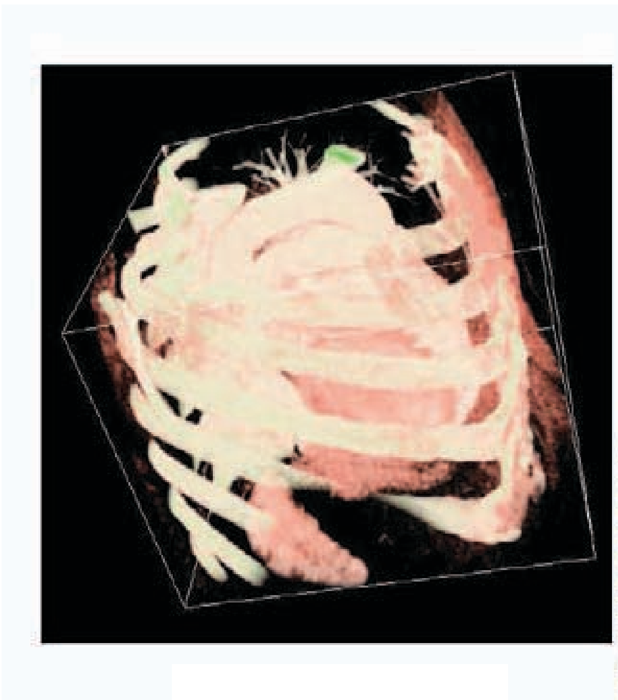
Irregular Hexahedral Mesh of Ventricles



Kroon, Technical Report, 2002

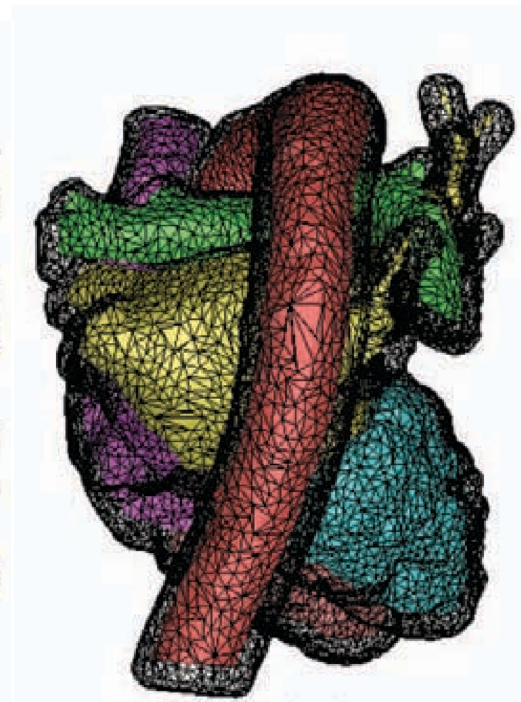
CVRTI

Generation of Irregular Meshes from Imaging Data



Computer tomography

Segmented image data



Tetrahedral mesh

Bajaj & Goswami, 2008

CVRTI

Group Work

Identify criteria for quality of meshes!

Compare regular with irregular meshes for applications in computational simulations of tissue electrophysiology!

Principle

Partial differential equation

- elliptical
- parabolic
- hyperbolic
- ...

Operators

- 1. Derivative spatial/temporal
- 2. Derivative spatial/temporal/mixed
- Grad / Div / Rot
- ...

Example

$$\alpha \frac{\partial u}{\partial t} + \beta \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\gamma \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial u}{\partial y}\right)$$

Approximation with differences

$$\frac{\partial u}{\partial t} \approx \frac{u_k - u_{k-1}}{\Delta t}$$

$$\frac{\partial^2 u}{\partial t^2} \approx \frac{u_{k+1} - 2u_k + u_{k-1}}{2\Delta t}$$

...

Compare with Euler-Method

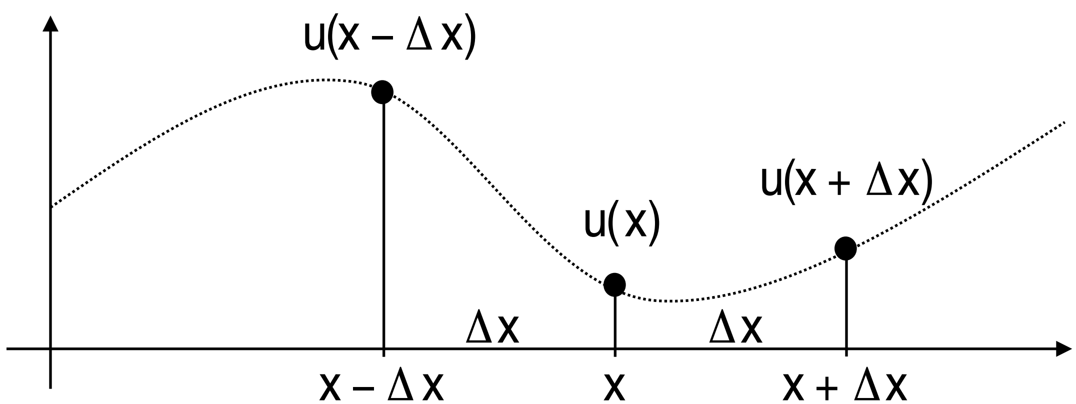
CVRTI

Discretization of 1D-Operators: 1st Spatial Derivative

Forward
$$u_x(x) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} \quad \rightarrow \quad u_x(k) = \frac{u(k+1) - u(k)}{\Delta x}$$

Backward
$$u_x(x) = \lim_{\Delta x \to 0} \frac{u(x) - u(x-\Delta x)}{\Delta x} \quad \rightarrow \quad u_x(k) = \frac{u(k) - u(k-1)}{\Delta x}$$

Central
$$u_x(x) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x} \quad \rightarrow \quad u_x(k) = \frac{u(k+1) - u(k-1)}{2\Delta x}$$

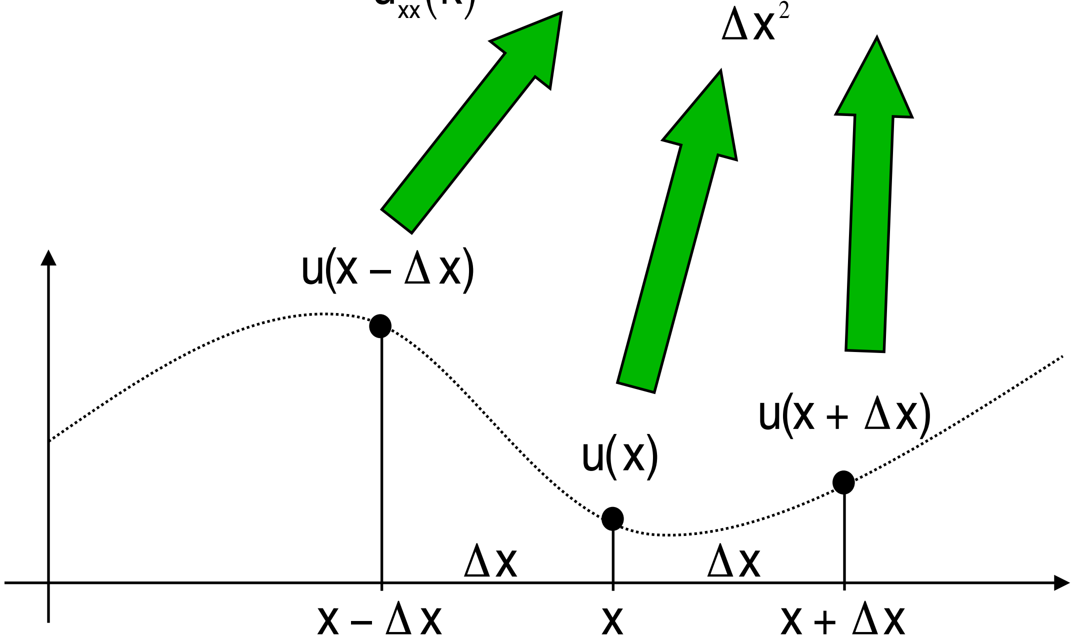


CVRTI

Discretization of 1D-Operators: 2nd Spatial Derivative

$$u_{xx}(k) = \frac{u_x\left(k+\frac{1}{2}\right) - u_x\left(k-\frac{1}{2}\right)}{\Delta x} \text{ mit } u_x(k) = \frac{u\left(k+\frac{1}{2}\right) - u\left(k-\frac{1}{2}\right)}{\Delta x}$$

$$\rightarrow \quad u_{xx}(k) = \frac{u(k+1) - 2u(k) + u(k-1)}{\Delta x^2}$$



Error of Finite Differences Approximation

Taylor series approximation

$$u(k \pm \Delta x) = u(k) \pm \frac{\partial u}{\partial x}(k)\frac{\Delta x}{1!} + \frac{\partial^2 u}{\partial x^2}(k)\frac{\Delta x^2}{2!} \pm \frac{\partial^3 u}{\partial x^3}(k)\frac{\Delta x^3}{3!} + \ldots$$

Forward difference

$$\frac{u(k+\Delta x) - u(k)}{\Delta x} = \frac{\partial u}{\partial x}(k) + \frac{\partial^2 u}{\partial x^2}(k)\frac{\Delta x}{2!} + \ldots = \frac{\partial u}{\partial x}(k) + E$$

$$\text{Error: } E = E(u, \Delta x) = \frac{\partial^2 u}{\partial x^2}(k)\frac{\Delta x}{2!} + \ldots$$

Central difference

$$\frac{u(k+\Delta x) - u(k-\Delta x)}{2\Delta x} = \frac{1}{2}\left(\frac{\partial u}{\partial x}(k) + \frac{\partial^3 u}{\partial x^3}(k)\frac{\Delta x^2}{3!} + \ldots\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x}(k) + E\right)$$

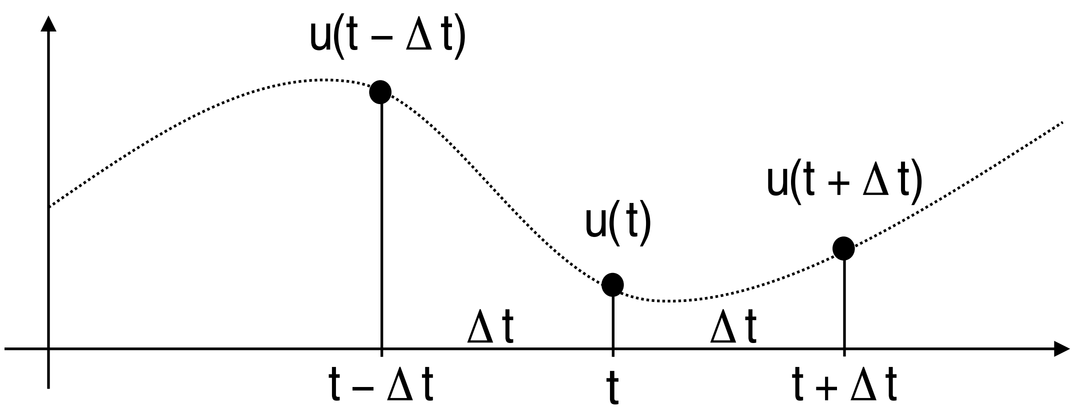
$$\text{Error: } E = E(u, \Delta x) = \frac{1}{2}\left(\frac{\partial^3 u}{\partial x^3}(k)\frac{\Delta x^2}{3!} + \ldots\right)$$

Discretization of 1D-Operators: 1st Temporal Derivative

Forward
$$u_t(x,t) = \lim_{\Delta t \to 0} \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} \rightarrow u_t(k,n) = \frac{u(k,n+1) - u(k,n)}{\Delta t}$$

Backward
$$u_t(x,t) = \lim_{\Delta t \to 0} \frac{u(x,t) - u(x,t-\Delta t)}{\Delta t} \rightarrow u_t(k,n) = \frac{u(k,n) - u(k,n-1)}{\Delta t}$$

Central
$$u_t(x,t) = \lim_{\Delta t \to 0} \frac{u(x,t+\Delta t) - u(x,t-\Delta t)}{2\Delta t} \rightarrow u_t(k,n) = \frac{u(k,n+1) - u(k,n-1)}{2\Delta t}$$

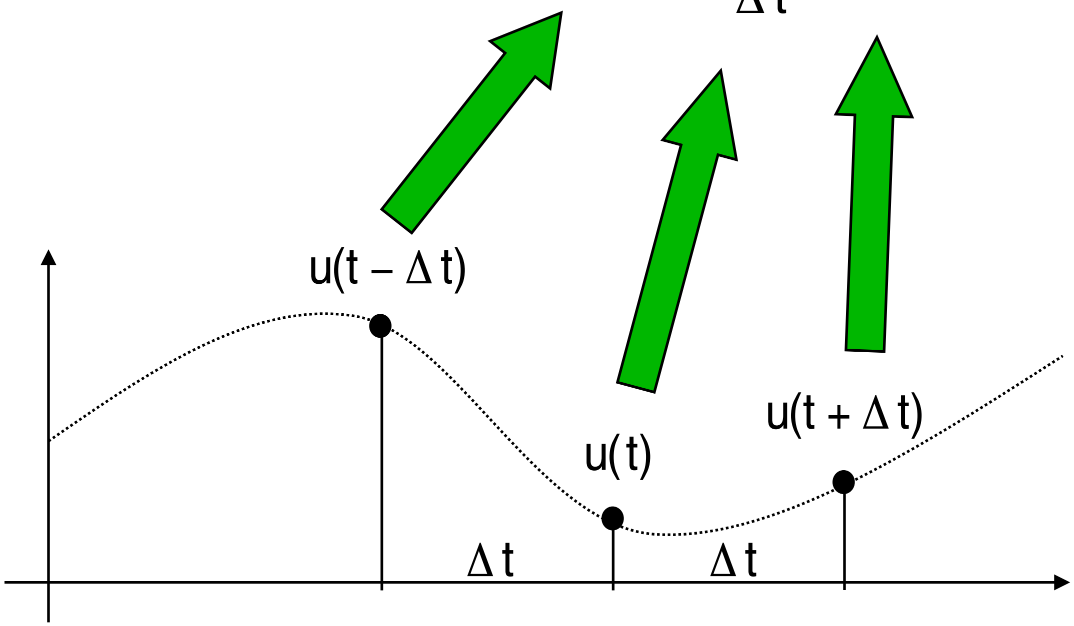


CVRTI

Discretization of 1D-Operators: 2nd Temporal Derivative

$$u_{tt}(k,n) = \frac{u_t\left(k,n+\tfrac{1}{2}\right) - u_t\left(k,n-\tfrac{1}{2}\right)}{\Delta t} \quad \text{mit} \quad u_t(k,n) = \frac{u\left(k,n+\tfrac{1}{2}\right) - u\left(k,n-\tfrac{1}{2}\right)}{\Delta t}$$

$$\rightarrow \quad u_{tt}(k,n) = \frac{u(k,n+1) - 2u(k,n) + u(k,n-1)}{\Delta t^2}$$



CVRTI

Discretization of 2D-Operators: 1st/2nd Spatial Derivative

$$u_x(x,y) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x-\Delta x, y)}{2\Delta x} \quad \to \quad u_x(k,j) = \frac{u(k+1,j) - u(k-1,j)}{2\Delta x}$$

$$u_y(x,y) = \lim_{\Delta x \to 0} \frac{u(x, y+\Delta y) - u(x, y-\Delta y)}{2\Delta y} \quad \to \quad u_y(k,j) = \frac{u(k,j+1) - u(k,j-1)}{2\Delta y}$$

$$u_{xx}(x,y) = \lim_{\Delta x \to 0} \frac{u_x\left(x+\frac{\Delta x}{2}, y\right) - u_x\left(x-\frac{\Delta x}{2}, y\right)}{\Delta x} \quad \to \quad u_{xx}(k,j) = \frac{u(k+1,j) - 2u(k,j) + u(k-1,j)}{\Delta x^2}$$

$$u_{yy}(x,y) = \lim_{\Delta y \to 0} \frac{u_y\left(x, y+\frac{\Delta y}{2}\right) - u_y\left(x, y-\frac{\Delta y}{2}\right)}{\Delta y} \quad \to \quad u_{yy}(k,j) = \frac{u(k,j+1) - 2u(k,j) + u(k,j-1)}{\Delta y^2}$$

$$u_{xy}(x,y) = \lim_{\Delta y \to 0} \frac{u_x\left(x, y+\frac{\Delta y}{2}\right) - u_x\left(x, y-\frac{\Delta y}{2}\right)}{\Delta y} \quad \to \quad u_{xy}(k,j) = \frac{u(k+1,j+1) - u(k-1,j+1) - u(k+1,j-1) + u(k-1,j-1)}{4\Delta x \Delta y}$$

Usage e.g. with 2D Poisson equation
Proceeding similar to discretization of mixed function u(x,t)

Discretization of 3D-Operators: div / grad of Scalar Functions

$$\nabla u(\vec{x}) = \begin{pmatrix} \dfrac{\partial u}{\partial x_1} \\ \dfrac{\partial u}{\partial x_2} \\ \dfrac{\partial u}{\partial x_3} \end{pmatrix} \rightarrow \nabla u(\vec{k}) = \begin{pmatrix} \dfrac{u(k_1+1,k_2,k_3) - u(k_1-1,k_2,k_3)}{2\Delta k_1} \\ \dfrac{u(k_1,k_2+1,k_3) - u(k_1,k_2-1,k_3)}{2\Delta k_2} \\ \dfrac{u(k_1,k_2,k_3+1) - u(k_1,k_2,k_3-1)}{2\Delta k_3} \end{pmatrix}$$

$$\nabla \cdot u(\vec{x}) = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3}$$

$$\rightarrow \nabla \cdot u(\vec{k}) = \frac{u(k_1+1,k_2,k_3) - u(k_1-1,k_2,k_3)}{2\Delta k_1} + \frac{u(k_1,k_2+1,k_3) - u(k_1,k_2-1,k_3)}{2\Delta k_2} + \frac{u(k_1,k_2,k_3+1) - u(k_1,k_2,k_3-1)}{2\Delta k_3}$$

Discretization of 1D Wave Equation with Central Differences

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

v: Velocity of wave propagation

$$u_{tt}(k,n) = v^2 u_{xx}(k,n)$$

$$\frac{u(k,n+1) - 2u(k,n) + u(k,n-1)}{\Delta t^2} = v^2 \frac{u(k+1,n) - 2u(k,n) + u(k-1,n)}{\Delta x^2}$$

$$\frac{u(k,n+1)}{\Delta t^2} = v^2 \frac{u(k+1,n) - 2u(k,n) + u(k-1,n)}{\Delta x^2} - \frac{u(k,n-1) - 2u(k,n)}{\Delta t^2}$$

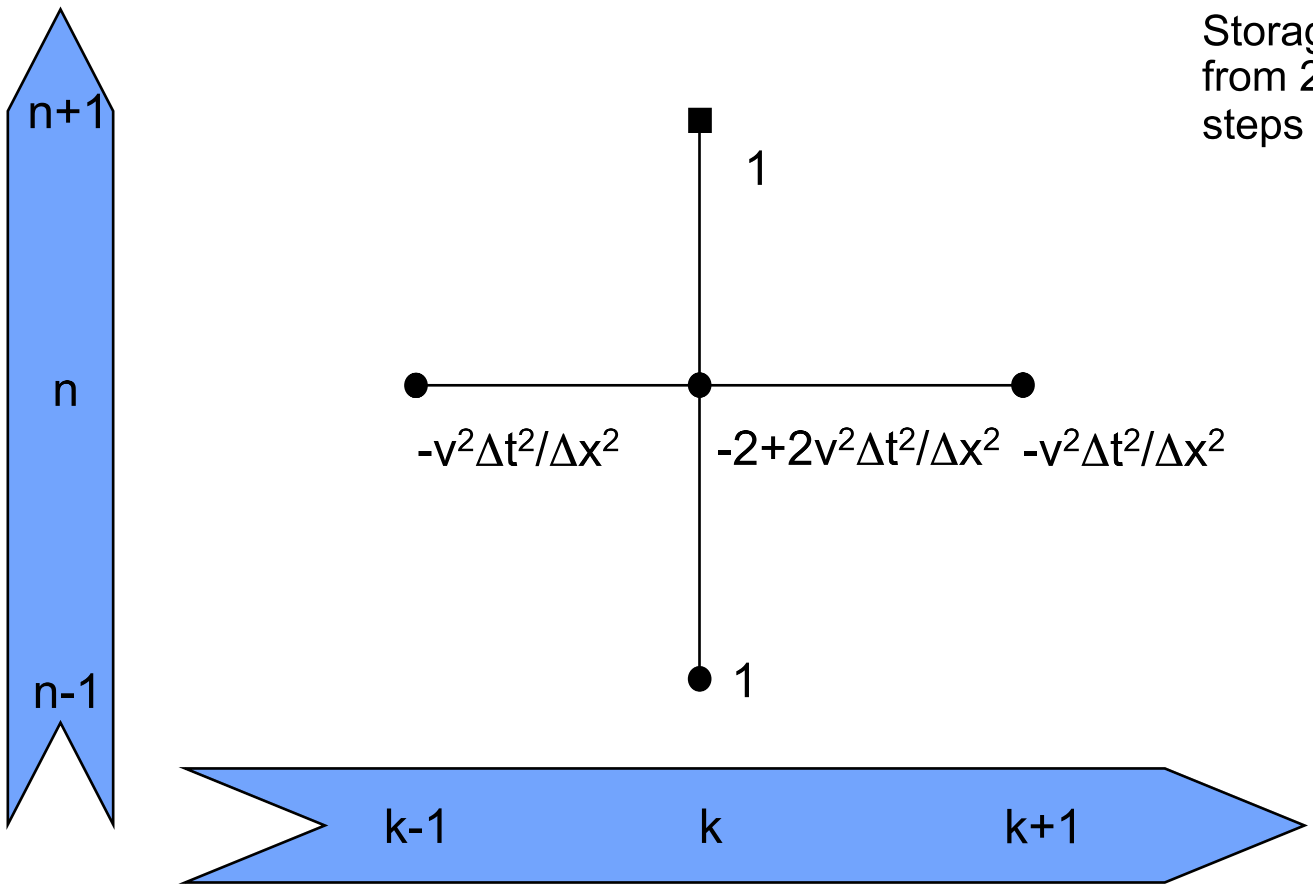
$$u(k,n+1) = \Delta t^2 v^2 \frac{u(k+1,n) - 2u(k,n) + u(k-1,n)}{\Delta x^2} - u(k,n-1) + 2u(k,n)$$

k: Spatial coordinate/index

n: Temporal coordinate/index

CVRTI

Schematic of 1D Wave Equation with Central Differences



Storage of node values from 2 previous time steps necessary!

CVRTI

Discretization of 1D Diffusion Equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right)$$

D: Diffusion coefficient

$$u_t(k,n) = D\, u_{xx}(k,n)$$

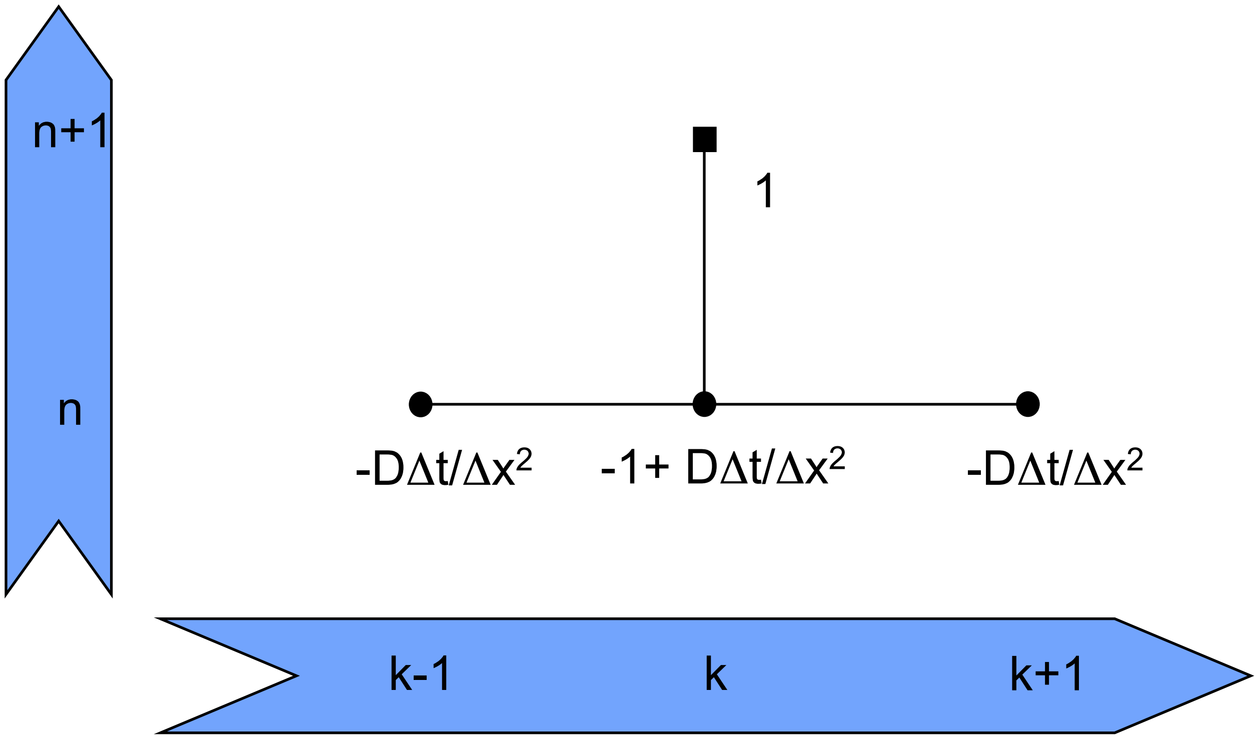
$$\frac{u(k,n) - u(k,n+1)}{\Delta t} = D\frac{u(k+1,n) - 2u(k,n) + u(k-1,n)}{\Delta x^2}$$

$$\frac{u(k,n+1)}{\Delta t} = D\frac{u(k+1,n) - 2u(k,n) + u(k-1,n)}{\Delta x^2} + \frac{u(k,n)}{\Delta t}$$

$$u(k,n+1) = \Delta t\, D\frac{u(k+1,n) - 2u(k,n) + u(k-1,n)}{\Delta x^2} + u(k,n)$$

CVRTI

Schematic of 1D Diffusion Equation



CVRTI

Discretization of 2D Poisson Equation

$$\rho(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$\rho(x,y)$: Source term

$$\rho(k,l) = u_{xx}(k,l) + u_{yy}(k,l)$$

$$\rho(k,l) = \frac{u(k+1,l) - 2u(k,l) + u(k-1,l)}{\Delta x^2} + \frac{u(k,l+1) - 2u(k,l) + u(k,l-1)}{\Delta y^2}$$

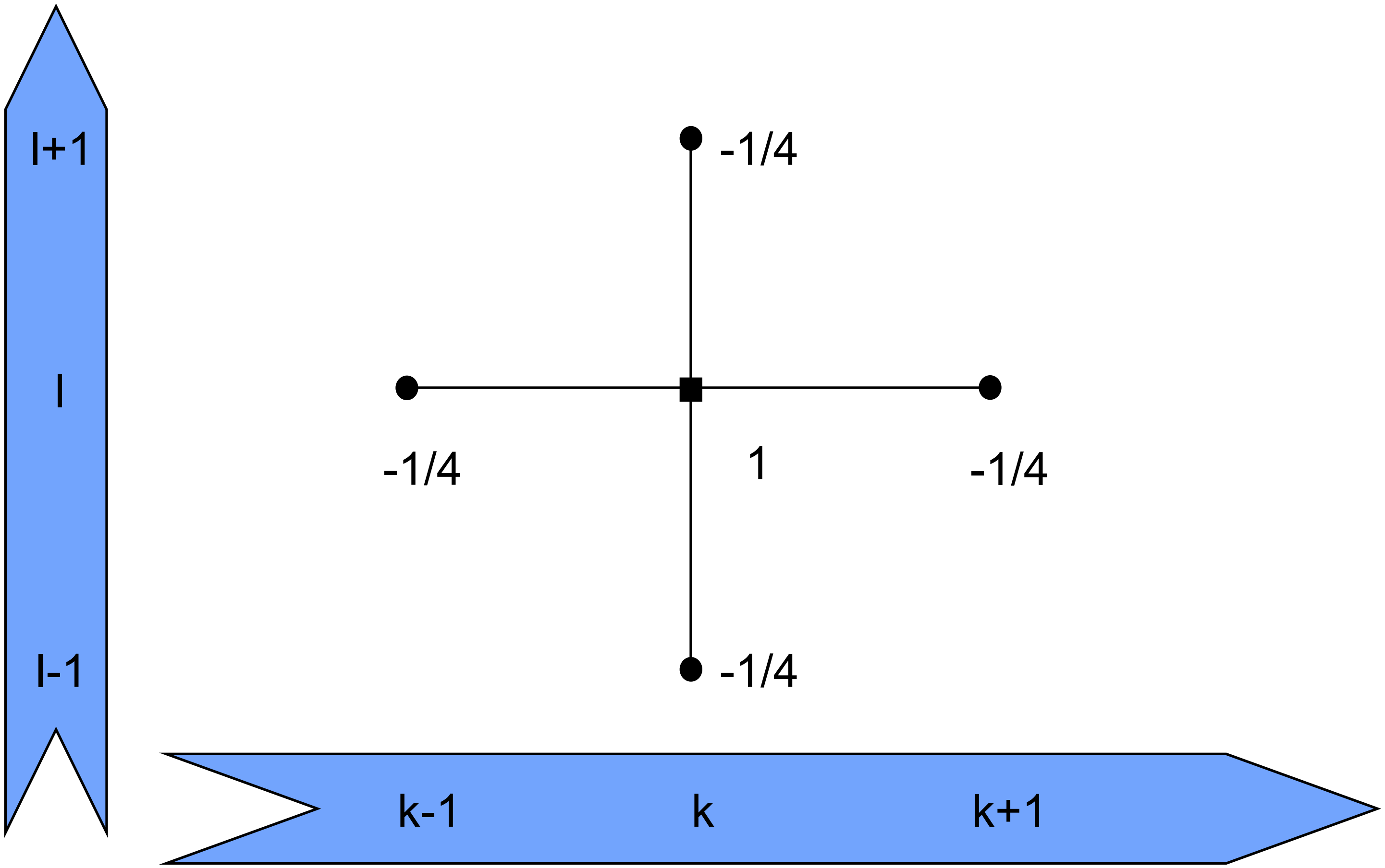
$$\frac{2u(k,l)}{\Delta x^2} + \frac{2u(k,l)}{\Delta y^2} = \frac{u(k+1,l) + u(k-1,l)}{\Delta x^2} + \frac{u(k,l+1) + u(k,l-1)}{\Delta y^2} - \rho(k,l)$$

$$\Delta x^2 = \Delta y^2 = \Delta^2$$

$$\rightarrow \quad u(k,l) = \frac{u(k+1,l) + u(k-1,l) + u(k,l+1) + u(k,l-1)}{4} - \frac{\Delta^2 \rho(k,l)}{4}$$

CVRTI

Schematic of 2D Poisson Equation



CVRTI

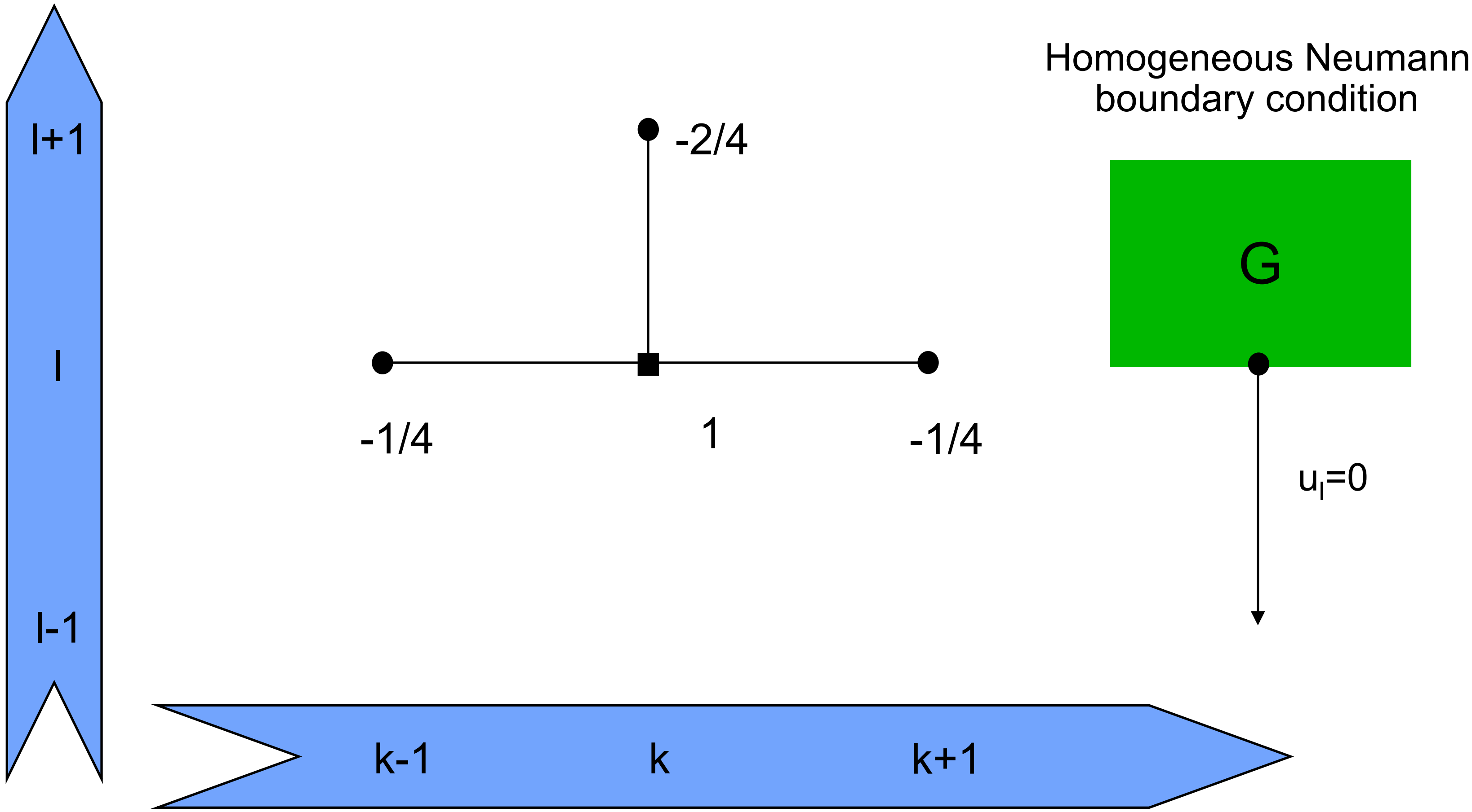
System Matrix For 2D Poisson Equation

$$
\begin{pmatrix}
M & M & M & M & M & M & M \\
 & & & -.25 & & & \\
 & & & -.25 & & & \\
 & -.25 & -.25 & 1 & -.25 & -.25 & \\
 & & & -.25 & & & \\
 & & & -.25 & & & \\
M & M & M & M & M & & M
\end{pmatrix}
\begin{pmatrix}
M \\ \phi_{k,l-1} \\ \phi_{k-1,l} \\ \phi_{k,l} \\ \phi_{k+1,l} \\ \phi_{k,l+1} \\ M
\end{pmatrix}
=
\begin{pmatrix}
M \\ M \\ M \\ -\dfrac{\Delta^2 \rho(k,l)}{4} \\ M \\ M \\ M
\end{pmatrix}
$$

- large dimension
- sparse
- banded
- symmetric
- positive semidefinite $\underset{\phi_s}{\forall} \quad \phi_s^T A_s \phi_s \geq 0$

CVRTI

Schematic of 2D Poisson Equation with Boundary Condition



CVRTI

Group Work

How is the approximation error controlled in the finite differences method?

CVRTI

Summary

- Partial Differential Equations
- Finite Differences Method
  - Discretization of Domains
  - Discretization of Operators
  - Discretization of Equations
- Summary

CVRTI